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Applications of Calculus in Economics: Marginal Function as an Example

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Abstract:

Mathematics is the cornerstone of life. It helps people to calculate, plan, predict and solve problems, enhances people's logical thinking and decision-making abilities, and is widely applied in finance, engineering, technology and other fields, providing strong support for daily life and career development. In addition, calculus is a core tool in modern science and engineering, used to study rates of change (derivatives) and accumulated quantities (integrals). It is indispensable in fields such as physics, economics, and engineering, providing a powerful framework for understanding and solving dynamic problems. This essay will examine the original and applications of calculus. To this end, in this paper the author will focus on how to use calculus to solve some problems in economics field. Besides, the author will also present examples to illustrate the application of calculus in economics, such as marginal function, price elasticity of demand, and so forth. This work highlights the importance of calculus in solving problems of economics.

Keywords: Calculus; Economics; Marginal function; Price elasticity of demand.

1. Introduction

Mathematics is a foundational subject for others and calculus have a significant and special value in this subject. When the theories of calculus were be found in 17th century by two different mathematicians, it was applied for a range of fields such as economics and physics [1]. Calculus involves function, limit, continuity, derivatives and integration. In calculus, people generally study two namely calculus, differential calculus and integral calculus. Most of physicists uses calculus frequently and perfectly such as motion, electricity, heat and light, astronomy, harmonic and other aspects of lives.

The concepts of derivatives and integration in calculus are the core foundation of optimization algorithms, which will be deeply applied to model parameter turning, neural network training, reinforcement learning and other fields in the future, promoting breakthroughs in AI in scenarios such as autonomous driving [2]. For example, meteorologists exploit calculus to imitate and forecast the change of the weather. In addition, in engineering design, calculus helps engineers calculate loads, pressures, and the stability of objects to ensure the safety and reliability of buildings and machinery. Then, in the medical field, calculus is used to analyze the speed of the spread of viruses and even plays an important role in drug discovery. However, compare with these fields, the relationship between economics and mathematics are more deepen and tighten in this generation [3]. For instance, quantitative economic analysis is a main method to develop and research the problems to find the optimal choices and maximize utilities for consumers, firms and so on. The application of calculus in option pricing and portfolio optimization will continue to deepen, and real-time risk warning and dynamic decision support in the financial market will be realized in combination with big data analysis.

Calculus is crucial in economics for analyzing functions and optimizing outcomes. It helps model economic phenomena like supply and demand, understand marginal effects, and solve optimization problems in production and consumption, providing insights for better decision-making [4]. This essay will discuss the origin of calculus in two theories and introduce applications in both macroeconomics and microeconomics. For microeconomics, it will study about the marginal function, elasticity analysis and optimal options etc. For macroeconomics, it will evaluate that how to use calculus to measure the Gross Domestic Products. In addition, this essay will take some applications of these knowledge.

2. Theory of Calculus

Tracking back to 17th century, Isaac Newton and Leibnitz were separately innovate calculus and come up with theories [5]. One of the most significant theory for calculus is 'Fluxions' and use '$\dot{x}$' to define velocity which invented by Newton when he studied of celestial motion. In the meantime, Leibnitz not only systematize the concept of calculus but also invent the symbol of derivation ($\frac{dy}{dx}$) and integration (∫).

The core of differential, to which the application of derivatives is an unlimited segment. The limit and continuity are basic conditions of derivatives [6]. Definition of limit is when independent variable towards to specific value, the value of function toward to constant value or infinitude. Sufficient and necessary conditions for left and right limits both existence and equality, i.e.,

$$\lim_{x\to x_0^-} f(x)=A \quad (1)$$

$$\lim_{x\to x_0^+} f(x)=A \quad (2)$$

$$\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)=A \quad (3)$$

So, $f(x)$ is smooth at x_o and $f'(x)$ exists.

The derivative is employed to depict the instantaneous rate of change at a specific point, which corresponds to the gradient at that point:

$$\lim_{\chi\to\chi_0}\frac{f(\chi)-f(\chi_0)}{\chi-\chi_0}=f'(\chi) \quad (4)$$

Furthermore, differentiation is the application of derivatives. It used to describe small instantaneous changes:

$$dy=y'dx \quad (5)$$

The core of integral, to which the inverse of the integral is an infinite summing. It describes the total amount of charge in a function over an integral. Definite and indefinite integrals are two different types of integral [7].

The concept of a definite integral is introduced when a function $f(x)$ is defined within a closed interval $[a,b]$, i.e.,

$$\int_a^b f(x)dx=\lim_{n\to 0}\sum_{i=1}^{r} f'(x)(\Delta x_i) \quad (6)$$

It uses to calculate the area of trapezoid. For illustration, see Fig. 1.

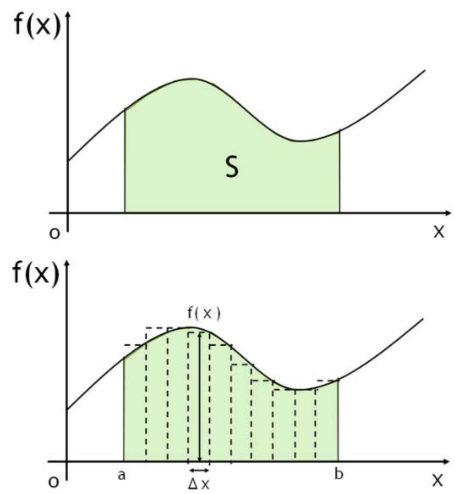

Fig. 1: illustration of the definite integral [7].

The definition of indefinite integral is that all original functions of the functions $f(x)$, i.e., $\int f(x)dx$. Assume y=f(x), indefinite integral of $f(x)$ regarding x is expressed by

$$\int f(x)dx+c \quad (7)$$

in which c is a constant. In addition, for definite integral it is calculated as $\int_a^b f(x)$. For example, it is calculated that

$$\int\left(x^3+x^2\right)dx=\frac{1}{4}x^4+\frac{1}{3}x^3+c \quad (8)$$

Below are a few relevant theorems.

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Theorem 1: Suppose the function $f(x)$ is continuous and differentiable. If $f'(x)>0$ for every x within the interval (a,b), then $f(x)$ is increasing on (a,b). Conversely, if $f'(x)<0$ for every x within the interval (a,b), then $f(x)$ is decreasing on (a,b).

Theorem 2 asserts that if a function $f(x)$ is continuous over the interval (a,b), any local extremum (either a maximum or a minimum) within this interval must be located at a critical point of $f(x)$.

Theorem 3: Suppose c is a critical value of the function $f(x)$. If the derivative $f'(x)$ is negative when $x<c$ and positive when $x>c$, then the point ($c, f(c)$) represents a local minimum. On the contrary, if f ′ (x) is positive for $x<c$ and negative for x>c, then the point $(c, f(c))$ is a local maximum.

Theorem 4: Let $f(x)$ be a differentiable function such that $f'(c)=0$. If $f'(c)>0$, then $f(x)$ attains a local minimum at the point (c,f(c)). Conversely, if $f''(c)<0$, then $f(x)$ attains a local maximum at $(c, f(c))$.

Theorem 5: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum and an absolute minimum on $[a,b]$. These absolute extremes of $f(x)$ will occur at a critical point or at an end point.

3. Applications

3.1 Marginal Function

The author shall first introduce the definition of Marginal Benefit. The marginal benefit associated with a good or service is defined as the incremental benefit obtained from producing an additional unit of said good or service [8].

Let $R=R(q)$ where q represents the outputs of goods or services, then $\frac{dR}{dP}=Q'(q)$ in which p represents the prices of goods or services. This is written as MR. The marginal cost is of producing one more unit of output. Let $C=C(q)$, then $\frac{dC}{dP}=C'(q)$, and this is written as MC [9].

For the marginal profit, it is associated with a good or service is defined as the incremental net revenue obtained from the production of an additional unit of that good or service. Let $L=L(q)=R(q)-C(q)$, then $\frac{dL}{dR}=R'(q)-C'(q)$, and it is written as ML. So, $\frac{dL}{dR}=\frac{dR}{dP}-\frac{dC}{dP}$, ML= MR- MC. Comparison of MR and MC is shown in Table 1.

Table 1: Comparison of MR and MC.

Conditions	Profit
MR> MC	positive for profit
MR< MC	negative for profit
MR= MC	maximum

There are many advantages of calculus in economic optimization. The calculus provides exact solutions rather than trial-and-error approach. There are several applications available.

Application 1. The entire cost (C) of generating a product is a function of output of the product (q):

$$C(q)=1.2q^2+5q-10 \tag{9}$$

The total revenue (R) of producing a product is a function of output of the product,

$$R(q)=0.5q^3+2q^2+2 \tag{10}$$

How to evaluate the marginal cost of the products when the output is 20 tonnes? To begin with, the author shall indicate the meaning of that representing. Here, $C'(q)=1.2\times 2q+5$ and $C'(20)=1.2\times 2\times 20+5=53$ (tonnes). Then, how to evaluate the marginal benefit of the products when the output is 20 tonnes? It is calculated as $R'(q)=0.5\times 3\times q^2+2\times 2\times q$ and $R'(20)=0.5\times 3\times 20^2+2\times 2\times 20=680$ (tonnes). Finally, how to calculate the marginal profit of the products when the output is 20 tonnes? It is $L'(q)=R'(q)-C'(q)=680-53=627$ (tonnes).

Application 2. The marginal revenue function is

$MR = 3 - 8x^2 + 3x$. The total revenue was given by the integral $\int 3 - 8x^2 + 3xdx = -16x^3 + 3 + c$, where c is a constant.

Application 3. A company sells bread for 5 dollars each and the cost associated with this bread is given by $C(x) = 2x + 0.5x^2 + 200$. The marginal cost is $C'(x) = 2 + x$ and $R(x) = 2x + 3$ with marginal revenue $R'(x) = 2$. $P'(x) = R'(x) - C'(x) = -x$. Thus, if one takes $x = 3$, then $C'(3)$ will represent the slope of the tangent line to the cost function $C(x)$ at a point $x = 3$.

3.2 Elasticity

Calculus can be used in price elasticity of demand and income elasticity [10]. It is used to measure of how sensitive demand is to changes in prices and incomes. One of the essential quantities is the price elasticity of demand (PED), meaning the responsiveness of changes in quantity demanded to changes in price. One of the methods to calculate it is by the ratio of percentage change in quantity demanded over percentage change in price, while the other method is

$$E(p) = \frac{-pf'(p)}{f(p)} \tag{11}$$

The classification of demand elasticity can be determined based on the magnitude of the elasticity value. Specifically, when the elasticity value exceeds 1, demand is characterized as price elastic; when the elasticity value is less than 1, demand is considered price inelastic; and when the elasticity value is precisely 1, demand exhibits unitary elasticity.

Application 4: judge the PED is elastic, inelastic or even unitary elasticity from price 6 to price 4? For the results shown in Table 2, it is found that

$$PED = \frac{changeindemanded}{changeinprice} = \frac{\frac{200-400}{200}}{\frac{6-4}{6}} = 3 \tag{12}$$

Since $3 > 1$, so demand is price elastic.

Table 2: Data for price and quantity.

price	8	7	6	5	4
quantity	0	100	200	300	400

Similarly, the income elasticity of demand (YED) serves as a quantitative metric to assess the responsiveness of demand for a good to changes in consumers' incomes. If the value of YED is ranged from -1 to 1, the demand income is elastic. If the value of YED is outside the above range, the demand income is inelastic. So, it can use these data to judge the type of goods whether belong to normal goods or inferior goods. Inferior goods are that an increasing in income will result in a fall in demand for other goods such as bread, canned tomatoes and reduce bus transport. In contrast, normal goods are that when their income increases, consumers will increase their demand for most goods.

Application 5. Let demand equation for a product be expressed as $x = f(p) = \sqrt{800 - 5p}$, and

$$E(p) = \frac{-pf'(p)}{f(p)} = \frac{-5p}{2(800-5p)} \tag{13}$$

When the price of the product is set at 50, the calculated elasticity of demand is found to be less than 1, thereby classifying the demand as inelastic. Under such conditions, an increase in the product's price is unlikely to result in a proportionate decrease in its demand.

Subsequently, the author will examine production profitability. Assuming an initial production volume q, one then proposes an output increment h. The marginal cost per unit of additional output h is given by:

$$A(h) = \frac{C(q+h) - C(q)}{h} \tag{14}$$

To evaluate the profitability of the production decision, a comparison with the market price is conducted. Consider a company with a second-order cost formula $C(q) = 0.001q^2 + 2q + 1500$, where q represents the output level. At an output level of $q = 1000$ and a market price $p = 5$, the total cost is calculated as $C(1000) = 4500$. Consequently, the average cost (AC) is derived as $AC = 1000C(1000) = 4.5$. Given that the market price exceeds the average cost, the production is deemed profitable. To further investigate the impact of increasing production, an increment h is introduced to the output level

$$A(h) = \frac{0.001(1000+h^2) + 2(1000+h)}{h} + \frac{1500-4500}{h} = 4 + 0.001h \tag{15}$$

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It is observed to be smaller than 5 when $h<1000$. Given the current market price, this implies that increasing the output level by h within this range would result in a significant enhancement of profitability.

As h approaches zero, mean cost of the added result is derived, representing the cost associated with producing an additional unit of output at the current production level q. This concept is closely related to the idea of a limit (refer to formula 2 for detailed derivation). The limit in this context is precisely the derivative of the cost formula, commonly referred to as the marginal cost

$$C'(q)=\lim_{h\to o}\frac{C(q+h)-C(q)}{h}. \quad (16)$$

Given the specified cost function, the marginal cost is calculated to be 4.

Application 6. The initial example examines the impact of an increased sales price on profit through the differentiation of the product functions. To ascertain the conditions of creating an increasing function, the revenue function is subjected to differentiation. Subsequently, the intervals where its first derivative exceeds zero are identified. Here, From the relation

$$\frac{dR}{dp}=\frac{dq}{dp}, p+q>0, \quad (17)$$

one finds that $-\frac{p}{q}\frac{dq}{dp}<1$. It is essential to note with price increasing relatively, the demand will correspondingly decrease by $-\frac{\text{\Deltaq}}{q}=E(p)\frac{\text{\Deltap}}{p}$. Typically, if $q(p)=120-2p$, thus $E(p)=\frac{2}{q(p)}p$. Further, if p : Price, q : Quantity, and R : Revenue are given, one can shows how the values of the variables of interest change in response to price fluctuations, based on a given demand function.

4. Conclusions

Although these examples of calculus applications in economics discussed in this paper represent only a small fraction of its potential uses in this field, they are sufficient to profoundly reveal the significant role that calculus plays in promoting the mathematical and quantitative analysis of economic phenomena. Calculus serves as a crucial tool for exploring economic laws and analysing economic phenomena. When applied appropriately, it can provide business decision - makers with precise data support, thereby offering objective and rational basis for corporate strategic decisions. Although the development of mathematics originates from economic needs, it now effectively serves the economic field and contributes to its efficient development.

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