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An Introduction to Eventown and Oddtown Problem

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Abstract:
The Eventown and Oddtown problem is a classic in Extremal Combinatorics, a field that examines set systems under specific constraints related to the relationships between sets and their elements. Tackling such problems is crucial as it involves decomposing complex issues into fundamental components, thus facilitating a transition from intuitive understanding to formal mathematical reasoning. Specifically, the Eventown problem centers on a scenario where 32 residents of a town aim to form groups, each containing an even number of members. The challenge lies in determining the maximum number of these groups. Rooted in abstract set theory and combinatorial techniques, this problem exemplifies the intricate connection between intuitive concepts and rigorous mathematical formalism, underscoring its significance in the realm of Extremal Combinatorics.

Keywords: Extremal Combinatorics, Set Theory, Combinatorial Optimization, Problem Solving, Mathematical Formalism

Introduction

This paper delves into the classical combinatorial challenge known as the Eventown and Oddtown problems, which scrutinize the formation of clubs under specific constraints concerning membership overlaps and club sizes. It begins with a detailed exploration of the Eventown problem, where each club must have an even number of members, and progressively integrates a pairwise intersection restriction to explore the broader dimensions of the problem. In contrast, the Oddtown problem is introduced, where all clubs are required to have odd cardinalities. To address these scenarios, the paper employs linear algebra and combinatorial proofs to define the upper and lower bounds for the possible number of clubs. These methods highlight the crucial role of incidence vectors and the properties of vector spaces in deriving solutions. The concluding sections revisit the Eventown problem, presenting alternative proof techniques and proposing potential areas for future research.

Extremal Set Theory offers a rich framework for dissecting the mathematical foundations of problems that are deceptively intuitive at first glance. The Eventown and Oddtown problems serve as quintessential examples in this field, demonstrating how restrictions on set cardinalities and intersections can lead to significant insights in combinatorial analysis. These problems fundamentally explore the limits on

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the number of subsets (or clubs) that can be established given specific rules, effectively linking abstract mathematical theory with practical applications in areas such as resource allocation and network design.
This paper starts with an exploration of the basic Eventown problem, requiring clubs to maintain even cardinalities. It addresses this challenge using bijections and combinatorial logic to present a solution. The discussion then progresses to a more comprehensive version of the Eventown problem that adds constraints on pairwise intersections between sets. In contrast, the Oddtown problem is introduced with a requirement for clubs to have odd cardinalities, which imposes a stark limitation on the possible number of clubs that can be formed. Employing techniques from linear algebra and combinatorial theory, we establish important results and bounds that underline the utility and sophistication of these mathematical approaches. The paper concludes by returning to the Eventown problem to examine alternative proof methods and suggest avenues for further scholarly exploration.

Overview of Extremal Combinatorics

Extremal Combinatorics is a branch of mathematical theory that focuses on determining the maximum or minimum number of subgroups that can be formed under various constraints. This field is particularly adept at tackling problems where a given set of objects is divided into subgroups with no overlap or under other specific conditions. For example, it can ascertain the largest subgroup size possible under preset rules. Extremal Set Theory, a closely related discipline, often comes into play in puzzle-solving scenarios where the logical leap from intuitive problem-solving to rigorous mathematical formalism is required. Although puzzles might appear straightforward, they are deeply ingrained with principles from abstract set theory and combinatorial reasoning.
A quintessential example of a problem in Extremal Set Theory is the Eventown problem. Consider a town with 32 residents who wish to form clubs. The number 32 is arbitrary, serving merely as a traditional placeholder in this theoretical construct. A key rule differentiates clubs: no two clubs may have the exact same membership. Moreover, an additional stipulation is that each club must have an even number of residents. The central question then becomes: What is the maximum number of distinct clubs that can be formed under these conditions?
To frame this question within Extremal Set Theory, we can assign labels to each resident (from 1 to 32) and to each potential club (from C1 to Cm, where m represents the total number of clubs). The constraints are as follows:
Ÿ Each club must be unique in composition compared to every other club ($C_i \neq C_j$ for any $1 \leq i \neq j \leq m$).
Ÿ The number of members in each club must be even ($|C_i|$ is even for any $1 \leq i \leq m$).
This setup not only highlights the applications of Extremal Combinatorics in theoretical puzzles but also underscores its relevance in abstract mathematical analysis and practical problem-solving.

Purpose of the Study

This study aims to determine the maximum number of clubs, denoted by m, that can be formed under specific conditions in the Eventown problem. To understand the uniqueness of each club, it's crucial to recognize that two clubs must differ by at least one resident. This implies that if two clubs are not identical, there must be at least one member who is included in one club but excluded from the other. Additionally, the cardinality of each club must be even, with the empty set also counted as a valid club containing no members.
This is the elementary version of the Eventown problem, and it is fairly easy to find the answer. In this case maxm = 231. In general, for n residents, the answer would be 2n−1. The idea is as follows.
If we ignore condition (2), then the problem essentially requires finding the number of subsets of $S := \{1,2,\cdots, 32\}$ because different subsets of S must have at least one element of difference. By set theory knowledge, we know that the number of subsets of S is given by the cardinality of the power set of S, i.e., P(S) = 232.
Adding the second condition only restricts the subsets of S to even cardinality. Therefore, we only need to find out how many subsets of S there are that contain an even number of elements, including the empty set. Intuitively, $\{A \subset S : A \text{ is even}\}$ should have an equal number of elements compared to $\{A \subset S : A \text{ is odd}\}$, and this is indeed true. We provide a combinatorics proof as follows.
Proof. The main idea is that we construct a bijection $f : \{A \subset S : A \text{ is even}\} \to \{A \subset S : A \text{ is odd}\}$ in order to show that the two sets have equal cardinality.
Take any element $a \in S$. Define:

$$f(A) = \begin{cases} A \cup \{a\} & \text{if } a \notin A \\ A \setminus \{a\} & \text{if } a \in A \end{cases}$$

This function is well-defined because, if a set is even, then adding or deleting an element would make it odd. If $A = \emptyset$, then we must have a ∈ / A so that f(A) = {a}. This is a bijection because f−1 clearly exists. Given an odd set A, we add a into it if a ∈ / A, and we delete it from A if $a \in A$. Thus, we are done. Since we are choosing $a \in A$ arbitrarily, the bijection between the two sets is not unique.
As a result, the maximum number of clubs that can be

formed is calculated to be ${}^{2}{}_{2}{}^{32} = 2^{31}$. This calculation confirms the substantial potential for forming a large number of distinct clubs in Eventown. Given this expansive possibility, the subsequent section of the paper will introduce an additional constraint to the Eventown problem. This new condition aims to further refine the structure of club formation and potentially reduce the maximum number of feasible clubs, thereby adding complexity and depth to our analysis.

The Full Eventown Problem

In the full Eventown system, a third condition exists. Each pair of clubs should share an even number of residents. Since 0 is an even number, we agree that two clubs sharing no residents also satisfy the condition.
In set theory, this means that: 1 ≤ i ≠ j ≤ m we have that |Ci ∩ Cj| is even.
Although this quantity would certainly be less than 2312^{31}231, it remains exponentially large. To illustrate, we can group the 32 residents into 16 pairs for simplicity, denoting these pairs as (1,2), (3,4) ,⋯ ,(31,32). Let $S^{'}$ = {(1,2),(3,4),⋯ ,(31,32)}. According to set theory, the collection of subsets of S′S'S′ has a cardinality of $2^{\frac{32}{2}} = 2^{16}$. Each subset of S′S'S′ can be considered a club, where all three conditions of the problem are naturally satisfied: specifically, the requirement that each pair of clubs shares an even number of residents is inherently met by the structure of S′.
If we initially assume there are n residents and n is odd, we can still generate a substantial number of clubs by modifying the previous scenario. Suppose $n > 1$ is odd. We first pick one resident and let him or her join no club. Notice that the remaining $n - 1$ residents are an even number, and thus we can construct $2^{\frac{n-1}{2}}$ clubs the same way as above. In summary, we see that, for a town of n residents, assuming $n > 1$, at least $2^{\lfloor \frac{n}{2} \rfloor}$ clubs can be formed. This lower bound suggests max$m \geq 2^{\lfloor \frac{n}{2} \rfloor}$, which is still exponentially large.
In the next section, we will modify one of the conditions to explore strategies for reducing the possible number of clubs that can be formed.

The Oddtown Problem

In this adjusted scenario, we impose a new condition: each club must share an odd number of residents with every other club. Observing a simple setup where each of the 32 residents joins exactly one unique club, we immediately find that there are 32 distinct clubs meeting conditions (1), (2*), and (3). Under this arrangement, each pair of clubs shares no residents, which also inherently satisfies condition (3), thus establishing that the minimum number of possible clubs, m is at least 32.
Surprisingly, attempts to further increase this lower bound are unsuccessful. However, using linear algebra techniques, we can establish that the upper limit for m is also 32. This demonstrates that the initial straightforward example not only meets all conditions but also represents the maximum configuration of clubs possible under these rules, though it is not the sole configuration that achieves this. We shall now give a proof of the upper bound.
The proof involves constructing a matrix where rows represent clubs and columns represent residents. A club includes a resident if and only if the corresponding matrix element is 1; otherwise, it is 0. The stipulation that each club must share an odd number of residents with every other club translates into ensuring that the dot product of any two different rows is odd. This condition imposes significant restrictions on the structure of the matrix, leading to a proof that no more than 32 such clubs can exist simultaneously without violating the constraints. The way we relate this problem to linear algebra begins by assigning each of the m clubs with an incidence vector. This means that, for each club Ci, we let Vi = (v1, v2, ⋯ , v32) where vj = 0 for j ∈ / Ci and vj = 1 for j ∈ Ci. In other words, the incidence vectors are vectors having 32 entries taking value from {0,1}.
The domain of these entries is the finite field F2. Given F_2^{32}, which contains all 32-tuples with elements in F2, we can define the addition and scalar multiplication in the most natural way as follows. For vectors $v = (v_1,v_2,\cdots ,v_{32})$,$u = (u_1,u_2,\cdots ,u_{32})$ and scalar λ ∈ F2.
Define:

$$\begin{aligned} v + u &:= (v_1 + u_1, v_2 + u_2, \cdots, v_{32} + u_{32}) \\ \lambda \cdot v &:= (\lambda v_1, \lambda v_2, \cdots, \lambda v_{32}). \end{aligned}$$

With abuse of notation, the operations on the right-hand side are addition and multiplications defined in the field F2. It is easy to check that $F^{3}{}_{2}{}^{2}$ under the operations defined above satisfies all conditions to become a vector space. Moreover, we can define the natural inner product on it by:

$$< u, v >:= \sum_{i=1}^{32} u_i v_i$$

To make the previous vector space an inner product space. The implications of the inner product are what matter. Notice that, when we relate the incidence vectors with the inner product operation, we have:
$< Vi,Vj >= |Ci \cap Cj|$ mod 2.
We have the mod 2 on the right-hand side because we are working in F2. In other words, saying that any two clubs share an even number of residents is equivalent to saying the inner product of the corresponding incidence vectors

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is 0. To summarize, if $C_1,\cdots,Cm$ satisfies condition (3), then we should have, for all $1 \le i,j \le m$.

$$< V_i, V_j >= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq \end{cases}$$

The essence of the proof is that, if we show all the incidence vectors are linearly independent, then we must have $m \le 32$ because we are working in $F^{3\ 2}_{2}$, i.e., the dimension of the vector space is 32. This provides an upper bound for the Oddtown problem.

Recall that, to show that $V_1,V_2,\cdots,V_{32}$ are linearly independent, we need to show that for $\lambda_1,\lambda_2,\cdots,\lambda_{32} \in F2$,

$\lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_{32} V_{32} = 0 \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{32} = 0$.

Suppose that $\lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_{32} V_{32} = 0$. Notice that if we multiply V_1 on each side we would have

$\lambda_1 < V_1,V_1 > + \lambda_2 < V_2,V_1 > + \cdots + \lambda_{32} < V_{32},V_1 > = 0$ $\lambda_1 \cdot 1 + 0 + 0 + \cdots + 0 = 0$ $\lambda_1 = 0$.

We can therefore multiply any Vi on both sides, which reduces the equation to $\lambda i = 0$. As a result, the incidence vectors are linearly independent, and we are done. □

Together with the lower bound provided at the beginning of the section, we have shown that max$m$ = 32. This is a satisfying result because we have brought down the maximum number of clubs to a linear quantity.

Theorem

In addition to utilizing linear algebra techniques, we can also approach Theorem (3.1) using a combinatorial framework. This combinatorial proof highlights why the Oddtown problem is not only foundational but also critically important in the study of Extremal Combinatorics. We prove this by way of contradiction. Suppose $n + 1$ clubs exist that satisfy the Oddtown system. We label them $C_1,C_2,\cdots,Cn_{+1}$. Let $F = \{C_1,C_2,\cdots,Cn_{+1}\}$. By set theory knowledge, we know that there are 2^{n+1} subsets of F. For any subset $A \subset F$ we define

$A^* = \{a \in \{1,2,\cdots,n\} : a$ is contained in an odd number of clubs in $A\}$.

In other words, A^* represents the set of residents who join an odd number of clubs in the collections of clubs labeled by A. There are at most 2^n numbers of such sets because there are only n residents. Therefore, by the Pigeonhole principle, there exists $A_1,A_2 \subset F$ such that $A^*_1 = A^*_2$.

Now, let B contain the elements that are in either A_1 and A_2 but not both. Notice that $B \subset F$ and that $B^* = \emptyset$. If, instead, there exists $a \in B^*$, then there exists an odd number of clubs that a is in. Since all of these clubs cannot be in both A_1 and A_2, an odd number of those clubs must be in A_1, and an even number of those clubs must be in A_2 or vice versa. Suppose that an odd number of those clubs are in A_1. We cannot merely say that$a \in A_1^*$ and$a \notin A_2^*$ because A_1 and A_2 might share an odd number of clubs that contains a, which will lead to $a \in A_1^*$ and$a \notin A_2^*$. However, regardless of the shared clubs between A_1,A_2 which are excluded from B, we always have$A_1^* \neq A_2^*$, which is a contradiction.

e that

$$\sum_{j=2}^{k} |C_{i_1} \cap C_{i_j}| \quad = \quad \sum_{a \in C_{i_1}} \sum_{j=2}^{k} |\{a\} \cap C_{i_j}|$$

Then, all residents are contained in an even number of clubs in B. Suppose that $B = \{Ci_1,Ci_2,\cdots,Cik\}$ where each Cij denotes a club and that B contains k clubs. Notic

$\equiv |Ci_1|$

The last deduction comes from the fact that $a \in Ci_1$ is contained in an even number of clubs in B, so it is contained in an$_k$odd number of clubs other than Ci_1. Since $|\{a\} \cap Cij|$ is either 0 or 1, the sum$\sum_{j=2} |\{a\} \cap C_{i_j}|$ is just counting the number of

clubs that a is in other than Ci_1, which is an odd number.

Notice that if the clubs $C_1,\cdots,Cn$ satisfies the Oddtown system, then by condition (2^*),$|Ci_1|$ must be an odd number. But by condition (3), each $|Ci_1 \cap Cij|$

should be even and hence$\sum_{j=2}^{k} |C_{i_1} \cap C_{i_j}|$ should be even.

This is a contradiction and we are done.

Conclusion

Throughout this paper, we have explored the intricacies of the Eventown and Oddtown problems within the context of Extremal Combinatorics. By employing both linear algebra and combinatorial proofs, we have demonstrated the maximum number of clubs possible under specific constraints. Our analysis conclusively established that the maximum number of clubs that can be formed is 32, effectively showing how additional conditions can significantly streamline and refine the problem space.

These findings not only reinforce the fundamental concepts of Extremal Set Theory but also illustrate the powerful interplay between mathematical intuition and rigorous formal analysis. The implications of this study extend beyond theoretical mathematics and offer insights into practical applications like network design and resource management, where understanding the limits of system configurations is crucial.

As we continue to delve deeper into the complexities of combinatorial mathematics, the lessons learned from the Eventown and Oddtown problems will undoubtedly serve as a valuable foundation for future research in the field. This work underscores the importance of theoretical exploration to uncover practical strategies that can be applied across various disciplines and real-world scenarios.

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