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Multi-Dimensional Catalan Numbers

Houzhi Wang[1, *],
Xihao Sha[2],
Jincheng Zhou[3]

[1]High School Affiliated to Renmin University of China, Beijing, 100086, P. R. China, wd23394@outlook.com
[2]High School Attached to Northeast Normal University, Changchun, 130000, P.R. China, shaxihao2024@outlook.com
[3]Shenzhen Huitong school, Shenzhen, 518000, P.R. China, Jerryzhou20080503@163.com

*Corresponding author email: wd23394@outlook.com

Abstract:
This article reviews the basics of the Catalan numbers and introduces a geometric proof for its formulae. Then, the work generalizes the Catalan numbers towards multi-dimensions. This paper later proposes the crucial steps in the foundation of the formulae on $C_n^{[a]}$ using Dyck paths, including defining $C_n^{[a]}$; introducing the concept of $T(m,n)$; finding and proving the relation between $C_n^{[a]}$ and $T(m,n)$ and proving the formulae on $T(m,n)$. Finally, this work solves a variant of the famous Ballot Problem by using the Hook length formula, pointing for future directions.

Keywords: Catalan numbers, Hook length formula, Bertrand's ballot problem, Dyck paths

1. Introduction

The Catalan numbers [1] were described in the 18th century by Leonhard Euler, who was interested in the number of different ways of dividing a polygon into triangles. The sequence is named after Eugène Charles Catalan, who discovered the connection to parenthesized expressions.

The most significant characteristic of the Catalan numbers is the recursive formula.

Denoted by C_n, the Catalan numbers are characterized by the following relation:

$$C_0 = 1, C_{n+1} = \sum_{i=0}^{n} C_i C_{n-1} (n \epsilon N)$$

In this article, we first reveal the key properties of the Catalan Numbers.

Previously many researchers have successfully found the generalization on Catalan Numbers in multi-dimensions [2][3][4], and we are here to conclude and push it further.

Finally, we will provide solutions to some probability problems related to the Betrand's Ballot Problem.

2. Geometric proof of the Catalan Numbers

Catalan numbers can be regarded as the number of lattice paths in a $(n \times n)$ grid from the origin to point (n,n) that stays below the diagonal [5].

We denote the term "good paths" as Dyck paths, and denote the term "bad paths" as the paths that cross the diagonal $y = x$. Let C be the set of "good

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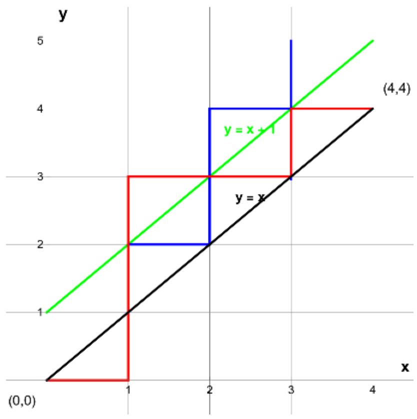
paths", and B as the set of "bad paths". It is obvious that all paths have the first step rightward. Therefore, $|B|+|C|$ $=\binom{2n-1}{n}$. We also denote D as the set of lattice paths in a $(n+1)\times(n-1)$ grid, and obviously $|D|=\binom{2n-1}{n+1}$



The figure on the right depicts random paths when $n=4$. The red path is a "bad path" for is passes the green line $y=x+1$.

After drawing several paths, we noticed that every "bad path" in B will have intersections with the line $y=x+1$. If we reflect the upper part of the "bad path" at the first intersection point, it can be shown that the reflected path is $(n+1)\times(n-1)$ and its first step is rightward. Therefore, we claimed that there exists a bijection between B and D.

It is obvious that 2 distinct paths in B gives distinct reflected paths. Thus, there exists an injection $f: B\to D$. Similarly, every path in D can be mapped to a path in B, so f is also surjective. Therefore, we get $|B| = |D|$.

In conclusion,

$$|C|=\binom{2n-1}{n}-\binom{2n-1}{n+1}$$

$$=\left(\binom{2n-1}{n}+\binom{2n-1}{n-1}\right)-\left(\binom{2n-1}{n+1}+\binom{2n-1}{n}\right)$$

$$=\binom{2n}{n}-\binom{2n}{n+1}=\binom{2n}{n}\left(1-\frac{n}{n+1}\right)$$

$$=\frac{1}{n+1}\binom{2n}{n}$$

3. Multi-dimensional Catalan Numbers

3.1 Impetus

We denote by $C_n^{[a]}$ the Catalan numbers in a dimensions. Naturally, the definition shall be the number of Dyck Paths $(0,\ldots,0)\to(n,\ldots,n)$ in a dimensions.

$C_n^{[a]}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$a=1$	1	1	1	1	1
$a=2$	1	2	5	14	42
$a=3$	1	5	42	462	6006
$a=4$	1	14	462	24024	1662804
$a=5$	1	42	6006	1662804	…………

Some Values of $C_n^{[a]}$

In the table it is noticeable that $C_n^{[a]}=C_a^{[n]}$, the symmetry occurs. To explain this behavior and obtain a general formula for it, we will introduce another sequence $T(m,n)$.

3.2 $T(m,n)$ and its relation to $C_n^{[a]}$

$T(m,n)$ [6], by definition, is the number of ways to arrange the integers $1mn$ in an $m\times n$ matrix L so that each row and each column is increasing. There are 2 things to notice: $T(m,n)$ is clearly symmetric, and it always equals

$C_n^{[m]}$, which we will see below.

To comprehend this equality intuitively, we see the entry $L_{ij} = k$ as "In the Dyck Path, the kth step is taken in the ith direction, and it is the jth step in this direction". It is easy to see that this forms a bijection.

Theorem.

$T(m,n) = C_n^{[m]}$

Proof by Bijection.

Define a mapping $f : M \to D$, where M is the set of $m \times n$ legal matrices and D is the set of Dyck Paths $(0,\ldots,0) \to (n,\ldots,n)$. Let $f(L)$ be the path in which the kth step is taken in the ith direction, and it is the jth step in this direction if the entry $L_{ij} = k$. Since all entries are distinct, there is no ambiguity; the increasing rows prevent wrong orders; the increasing columns say that the $(i+1)$ th direction mustn't exceed the ith.

f is injective because distinct matrices L_1 and L_2 must lead to distinct paths of differenct order of steps taken.

f is surjective because clearly f^{-1} can be defined for any Dyck Path to generate a matrix.

Thus, f is a bijection, for which $|M| = |D|$.

?

Theorem.

$$T(m,n) = \frac{(mn)!sf(m-1)sf(n-1)}{sf(m+n-1)}$$

where $sf(x)$ denotes the super factorials, $\prod_{i=1}^{x} i!$.

Proof.

The definition of our "legal" matrices coincides with the definition of standard Young tableaux.

By the Hook Length Formula [7],

$$T(m,n) = f^{\lambda} = \frac{(mn)!}{\prod_{c\in\lambda} h_\lambda(c)}$$

where λ is the Young Diagram of shape $m \times n$ and $h_\lambda(c)$ is the hook length of the cell c.

Consequently,

$$T(m,n) = (mn)!\left(\prod_{i=1}^{m}\prod_{j=1}^{n} m+n-i-j+1\right)^{-1}$$

$$= (mn)!\left(\prod_{i=1}^{m}\prod_{j=1}^{n} i+j-1\right)^{-1}$$

$$= (mn)!\left(\frac{n!}{0!}\frac{(n+1)!}{1!}\ldots\frac{(n+m-1)!}{(m-1)!}\right)^{-1}$$

$$= \frac{(mn)!sf(m-1)}{n!\ldots(n+m-1)!}$$

$$= \frac{(mn)!sf(m-1)sf(n-1)}{sf(m+n-1)}$$

?

An elementary proof of the Hook Length Formula can be found at [8].

3.3 Approximation

By the Stirling's Approximation formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we may look closer on the growth rate of the Multi-dimensional Catalan numbers.

$$sf(n) \approx \prod_{i=1}^{n} \sqrt{2\pi i}\left(\frac{i}{e}\right)^i = (2\pi)^{\frac{n}{2}} exp\left(-\frac{n(n+1)}{2}\right)\prod_{i=1}^{n} i^{i+\frac{1}{2}}$$

is clear.

Then

$$C_n^{[m]} \approx$$

$$\sqrt{2\pi mn}\left(\frac{mn}{e}\right)^{mn}(2\pi)^{-\frac{1}{2}}$$

$$exp\left(\frac{(m+n)(m+n-1)-m(m-1)-n(n-1)}{2}\right)\frac{\prod_{i=1}^{m-1} i^{i+\frac{1}{2}}\prod_{i=1}^{n-1} i^{i+\frac{1}{2}}}{\prod_{i=1}^{m+n-1} i^{i+\frac{1}{2}}}$$

$$= (mn)^{mn+\frac{1}{2}}\frac{\prod_{i=1}^{m-1} i^{i+\frac{1}{2}}\prod_{i=1}^{n-1} i^{i+\frac{1}{2}}}{\prod_{i=1}^{m+n-1} i^{i+\frac{1}{2}}}$$

The fraction on the right is almost always less than 1, so we can tell

$$C_n^{[m]} \ll (mn)^{mn}$$

when both m and n are sufficiently large, but further studies are required to locate its growth more precisely.

4. Probability Problems

Bertrand's Ballot Problem [9] is stated as "In an election where candidate A receives p votes and candidate B receives q votes with p > q, what is the probability that A will be strictly ahead of B throughout the count? (Bertrand, 1887)"

But since we are interested in the scenario where $p = q$, in which the problem always ends up with the boring probability 0, we study the variant "A will not fall behind B" [10] instead.

Denote this probability $\mathcal{P}(p)$.

The total number of possible processes is clearly equivalent to the number of lattice paths $(0,0) \to (n,n)$, which

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is $(\genfrac0pt2nn)$, denoted by D_p.

$$\mathcal{P}(p)=\frac{C_p}{D_p}=\frac{1}{p+1}$$

is obvious.

Naturally we generalize this to other dimensions by denoting the number of lattice paths in a dimensions by $D_p^{[a]}$ and the probability of the vote counts of the a candidate always forms a non-increasing sequence by $\mathcal{P}^{[\alpha]}(p)$.

With the same idea,

$$D_p^{[a]}=\prod_{i=1}^{a}(\genfrac0ptipp)=\frac{p!}{p!0!}\frac{(2p)!}{p!p!}\cdots\frac{(ap)!}{p!((a-1)p)!}=\frac{(ap)!}{p!^a}$$

$$\mathcal{P}^{[a]}(p)=\frac{C_p^{[a]}}{D_p^{[a]}}=\frac{sf(a-1)sf(p-1)p!^a}{sf(a+p-1)}=\frac{sf(a-1)sf(p)p!^{a-1}}{sf(a+p-1)}$$

This looks unpleasant, but observe that:

$$\mathcal{P}^{[2]}(p)=\frac{1}{p+1}$$

$$\mathcal{P}^{[3]}(p)=\frac{2}{(p+1)^2(p+2)}$$

$$\mathcal{P}^{[4]}(p)=\frac{12}{(p+1)^3(p+2)^2(p+1)}$$

…

a better formula can be found:

$$\mathcal{P}^{[a]}(p)$$

$$=\frac{sf(a-1)sf(p)}{sf(p)}\cdot\frac{p!^{a-1}}{\prod_{i=p+1}^{a+p-1}i!}$$

$$=sf(a-1)\cdot\prod_{i=p+1}^{a+p-1}\frac{p!}{i!}$$

$$=sf(a-1)\cdot\prod_{i=p+1}^{a+p-1}\prod_{j=p+1}^{i}j$$

$$=sf(a-1)\cdot\prod_{i=p+1}^{a+p-1}i^{a+p-i}$$

$$=\frac{sf(a-1)}{\prod_{i=1}^{a-1}(p+i)^{a-i}}$$

Here is some additional information:

$$\mathcal{P}^{[\alpha]}(1)=\frac{1}{a},$$

And

$$\forall a,\lim_{p\to\infty}\mathcal{P}^{[\alpha]}(p)=0.$$

They can be observed and shown easily.

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