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Variance Reduction in Monte Carlo Option Pricing: A Comparative Analysis of Control Variates, Multiple Control Variates and Antithetic Variates

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Abstract:

Monte-Carlo Simulation is a method to solve many financial problems but the convergence rate of the simple Monte-Carlo method tends to be low. In Monte-Carlo Simulation, variance reduction is a technique to reduce the accuracy of the simulation in order to increase the efficiency and accuracy of the simulation. This paper aims to investigate and compare the methods which reduce the variance for the Monte Carlo simulation. Three variance reduction techniques, involving control variates, multiple control variates and antithetic variates, are implemented to Monte Carlo simulation for option pricing, including European and Power put and call options. Multiple control variates is a linear combination of variates in control variates. Black-Scholes Model are used as a benchmark for the validation of the accuracy and effectiveness of three methods in an initial condition. Various scenarios, compromising different Strike Prices, Risk-Free Interest Rates and Volatilities, are used to analyse their impact on the performance of three methods and to compare the effectiveness of these techniques in different occasions. Sufficient simulation paths are set to be 1 million, which ensures statistical stability. The results demonstrate that multiple control variates perform as the best technique in most cases, while in some cases antithetic shows a better performance than the other 2 methods. Moreover, control variates sometimes give a similar effectiveness in variance reduction compared to multiple control variates, but it performs worse in most experimental occasions than multiple control variates.

Keywords: Monte Carlo Simulation, Option Pricing, Variance Reduction, Control Variates, Antithetic Variates, Multiple Control Variates, Black-Scholes Model

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1. Introduction

With the rapid development of today's world, financial markets have also changed dramatically. In order to satisfy various trading needs, financial derivatives were born. The complexity of financial markets leads to increased trading risks, derivatives are inherently risk-averse, it has development rapidly in recent years. Financial derivatives are financial products derived from underlying assets such as currencies, stocks, bonds, commodities or something like that, which the value depend on the subject matter, and can be infinitely derived on this basis [1]. According to the statistics, the total value of all financial derivatives in the world reached 1000 trillion US dollars at its peak points, which far exceed the total global assets.

In 2008, the US subprime mortgage crisis triggered a global financial crisis, which severely hampered global economic development and caused some major economies to fall into recession [2]. The financial markets of some developed countries also suffered heavy losses in this financial crisis, with the US, which has the most developed financial system and financial derivatives in the world, suffering the most serious losses. According to investigations, the main cause of this crisis was the globalization of financial derivatives and the lack of financial supervision in the field of financial derivatives [1].

After the epidemic, the economy continued to recover, and artificial intelligence ushered in unprecedented rapid development. Shares of semiconductor companies such as Nvidia and AMD continued to rise, pushing the Nasdaq index to new highs. In Europe, Germany's DAX and Japan's Nikkei also continued to hit record highs. Against the backdrop of optimistic market sentiment and rising stock prices, options, as a relatively flexible contract system, have become a favored financial derivative among many investors, such as European options. At present, more and more companies and individuals want to invest in financial derivatives, but because of the risks of these products, they will have more decisions to hold these derivatives. Therefore, for financial derivatives such as options, reasonable and accurate option pricing has become one of the topics discussed in this paper.

The Black-Scholes model is a mathematical formula used to calculate option pricing in financial derivatives. The application of this model is to eliminate potential risks and obtain arbitrage opportunities by buying and selling unreasonable options and hedging the assets held at the same time. This method is also called "dynamic Delta neutrality". However, this model has limitations. It is observed in the market that the implied volatility of options is usually not a constant, which is inconsistent with the assumptions of the BS model and leads to the volatility smile phenomenon. However, it assumes that price changes conform to the normal distribution. If statistical fat tails frequently occur in the financial market, it will affect the effectiveness of the BS model formula [3].

The Monte Carlo method is a random simulation method based on probability theory and statistical theory. It is a computational method that uses random numbers or more commonly pseudorandom numbers to solve many computational problems. The desired problem is combined with the relevant probability model, and the high school sampling is simulated by computer to obtain an approximate result of the problem. The Monte Carlo method is also widely used in the field of financial engineering. Monte Carlo option pricing is a commonly used method in financial mathematics [4]. The Monte Carlo method obtains a large number of possible asset price paths through random sampling, and then performs statistical analysis on the option values on these paths to finally obtain the option price.

Based on the advantages of each model, Monte Carlo simulation can deal with various options pricing problems flexibly and efficiently. The BS model is commonly used in standard European option pricing and provides a theoretical framework and benchmark. By combining Monte Carlo simulations with Black-Scholes models, more complex European-style option pricing or risk management can be handled flexibly and efficiently.

This method still has certain limitations. In option pricing, option value is essentially the expected value of certain random variables. Therefore, Monte Carlo simulation can be used to approximate this expected value by calculating a large number of samples. The estimation of Monte Carlo method is accompanied by certain errors, and a significant feature of this method is its slow convergence speed, which may lead to an increase in the variance of the estimated price. Therefore, how to reduce variance and improve the accuracy of Monte Carlo methods has become the main topic of this article. There are currently various techniques for reducing variance, such as control variates, antithetic variates and so on. The control variates is a method to reduce the variance of Monte Carlo simulation estimates. This method introduces an auxiliary variable Y with a strong correlation and a known expected value, which can significantly reduce the variance of the original estimator. If the random variable Similarly, the antithetic variates method is also a technique used to reduce the variance of Monte Carlo simulation estimates. This method uses antithetic random variables included two opposite variables to eliminate the dependence between simulation paths. We must not only consider the objective function, and also consider the value of opposite objective function, so that we take advantage of the average symmetry of the sample to reduce the variance, thereby effectively improving the accuracy of Monte Carlo simulation. In this article we will further explore the control variates method to reduce variance, and which parameters will effectively reduce variance, and compare it with the antithetic variates method.

2. Literature Review

Monte Carlo simulation is a widely used method for calculating complex option pricing in the fields of financial engineering and derivative pricing. Monte Carlo simulation has good robustness, but it still generates significant variance during the calculation process, leading to inaccurate option pricing and errors, which affect investment and trading. In order to reduce the variance of Monte Carlo simulation, this paper further explores and improves the control variates and antithetic variates.

The basic principle of the control variates method is to reduce the variance by introducing auxiliary variables with known expected values, that is, to improve the accuracy and efficiency of simulation results by selecting appropriate control variates [5]. For example, [6] introduced the theoretical basis and application methods of the control variates method in detail, emphasizing the importance of selecting control variates and how to optimize the application of control variates. In addition, [7] and [8] mentioned that the application of the control variates method in option pricing has great potential. Such as American option pricing, the control variates method was united in wedlock other variance reduction techniques, which can effectively work out highdimensional and path-dependent problems [7] and [8].

The variance in the calculation process can be reduced by antithetic variates, usually by constructing a antithetic variables that is negatively correlated with the original random variable and generating a pair of sample paths that cancel each other's variances. This method ensures that the mean of the pair of samples provides a more accurate estimate values and reduces the standard variances[9]. [10] showed how to use a antithetic approach to fit the volatility smile in a jump-diffusion process and achieve an accurate simulation of option prices.

Latest improvements in control variables, including multiple control variates[11]. Researchers found that the multiple control variates was pretty effective when the several control vairates achieved in Monte Carlo and quasi-Monte Carlo methods. These researches emphasize the adaptability and robustness of multiple control variates in reducing variance, which create a new research direction for more accurate option pricing.

[12] highlight the reducing variances and optimizing transaction costs for option pricing is important for Monte Carlo simulation. Specifically, the study further explore the application of quasi-Monte Carlo methods and their effectiveness in reducing the effective dimension, which directly affects the accuracy of derivative pricing models [13]. Referring to the research, the intrinsic value and external volatility are the key factories for the Monte Carlo simulation accuracy.

Overall, the variance reduction techniques, included control variates, antithetic variates and multicontrol variates play a vital role in promoting financial modeling development and option pricing. Understanding and effectively applying these variance reduction techniques can significantly improve the accuracy and efficiency of Monte-Carlo simulation, have shown great potential and universal applicability in improving the accuracy of Monte Carlo simulations.

3. Methodology

3.1 Research Question

The main research question is: How effective are control variates, multiple control variates, and antithetic variates in reducing variance in Monte Carlo simulations for option pricing, and how do key parameters such as the strike price (K), risk-free interest rate (r), and volatility (σ) influence these effects? This research aims to improve the accuracy and efficiency of Monte Carlo methods in financial modeling by experimentally comparing these variance reduction techniques through different market conditions. Specifically, the study aims to investigate:

- RQ1: How does the choice of strike price (K) impact the effectiveness of control variates, multiple control variates, and antithetic variates in variance reduction?
- RQ2: How does the risk-free interest rate (r) and volatility (σ) affect the effectiveness?
- RQ3: How do these 3 techniques differ in their ability to reduce variance in Monte Carlo simulations under these market conditions?

3.2 Research Design

This study applies a quantitative experimental design to evaluate and compare the effectiveness of control variates, multiple control variates, and antithetic variates in reducing variance during Monte Carlo simulations for option pricing. The Black-Scholes (BS) model serves as the benchmark for theoretical option prices, providing a reference point to assess the accuracy of the Monte Carlo estimates.

The research design includes the following steps:

1. Simulation Setup: Conducting Monte Carlo simulations

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for both European and power put and call options with various parameters such as strike price (K), risk-free interest rate (r), and volatility (σ). Then control variates, multiple control variates, and antithetic variates techniques are implemented and their effectiveness in variance reduction are evaluated.
2. Validation with Black-Scholes Model: The option prices, computing with the basic parameter values, generated from the Monte Carlo simulations are compared to the theoretical prices obtained from the Black-Scholes model as a validation of their accuracy.
3. Variance Analysis: The variance of each Monte Carlo simulation is computed and the variances with same conditions result in a score judging the effectiveness of variance reduction.
4. Parameter Sensitivity and Comparison: Sensitivity analysis is performed to explore how changes in strike price (K), risk-free interest rate (r), and volatility (σ) affect the variance reduction achieved by each technique. In addition, the effectiveness of each technique is compared.
This design enables a comprehensive evaluation of how control variates, multiple control variates, and antithetic variates influence variance reduction in option pricing and their effectiveness under different conditions.

3.3 Variables Used

In this study, the following variables are used for analysis:
• Dependent Variable:
o Variance of Monte Carlo Simulations: The variance in the estimated option prices obtained from Monte Carlo simulations.
• Independent Variables:
o Strike Price (K): The strike price of the options ranging from 0 to 300 to analyse the influence on variance reduction using the following techiniques, control variates, multiple control variates, and antithetic variates.
o Risk-Free Interest Rate (r): The risk-free interest rate ranging from 0 to 0.10.
o Volatility (σ): The volatility of the underlying asset ranging from 0 to 0.5.
o Control Variates (Y_1, Y_2): Two control variates chosen based on their correlation with the option price, including the underlying asset's final price and its logarithm.
o Multiple Control Variates (Y): Combining Y_1 and Y_2 linearly, using the method in [11].
o Antithetic Variates: Using of antithetic pairs (Z and $-Z$) to reduce variance.
• Control Variables:
o Number of Simulation Paths (N): The number of paths generated in the Monte Carlo simulation, kept constant at 10 million paths during each set of experiments to ensure statistical stability.
o Constant Market Parameters: The initial stock price (S_0) is kept constant at 100 and the maturity time (T) and the trading days are kept at 1 year and 52 weeks respectively to focus on the impact of K, r, and σ on variance reduction.

3.4 Method of Data Collection

This research applies Monte Carlo simulations to model the price paths of the underlying asset and evaluate the effect of control variates, multiple control variates, and antithetic variates on variance reduction in option pricing. Data collection involves following steps:
1. Simulation Setup: The underlying asset prices are simulated using a geometric Brownian motion model. The parameters include an initial asset price (S_0) of 100, a risk-free interest rate (r) of 0.1, a volatility (σ) of 0.3.
2. Monte Carlo Simulation and Variance Reduction Evaluation: The simulation is conducted by discretizing the time horizon into 52 weekly intervals, and each step in the asset price is calculated using the following equation:

$$S_t = S_{t-1} exp\left(\left(r - 0.5\sigma^2\right)dt + \sigma\sqrt{dt}Z\right)$$

where Z is a standard normal random variable, and dt is the length of the time step. The effectiveness of control variates, multiple control variates, and antithetic variates in reducing variance is evaluated by the following formula:

$$1 - \frac{\mathrm{Var}\left(\bar{X}_n\left(b^*\right)\right)}{\mathrm{Var}\left(\bar{X}_n\right)}$$

where $\mathrm{Var}\left(\bar{X}_n\left(b^*\right)\right)$ is the variance of simulation with a technique and $\mathrm{Var}\left(\bar{X}_n\right)$ is that of simple Monte Carlo simulation.
3. Analysis and Comparison: Analysing the variance reduction effectiveness of control variates, multiple control variates, and antithetic variates with different K, r, and σ.
Hurdles, Challenges and Amendments
During the data collection process, several challenges were encountered:
• Computational Resources: 10 million paths were a relatively large number for CPU and we used GPU computing to accelerate so that we can complete our work efficiently.
• Sample Size Determination: 10 million paths were sufficient to ensure statistical stability and with GPU computing, a good balance has been reached between accuracy

and efficiency.

3.5 Conceptual Framework

This study is based on the theoretical foundations of option pricing and variance reduction techniques. The core concepts involve the Black-Scholes (BS) model [14] as a benchmark for evaluating option prices, control variates [15], multiple control variates combining Y_1 and Y_2 [16] and antithetic variates [15] as techniques reducing the variances of simulations.

4. Results

4.1 Validation with Black-Scholes Model

To ensure the validity of our Monte Carlo simulations, we compared the results to the theoretical option prices provided by the Black-Scholes (BS) model under the baseline market conditions:

$S_0 = 100, ?K = 100, ?T = 1, ?r = 0.1, ?\sigma = 0.3$

Figure 1 shows a comparison between the Monte Carlo estimates (with and without variance reduction techniques) and the BS model prices for call option. As expected, the variance reduction techniques bring the Monte Carlo estimates closer to the BS theoretical values.

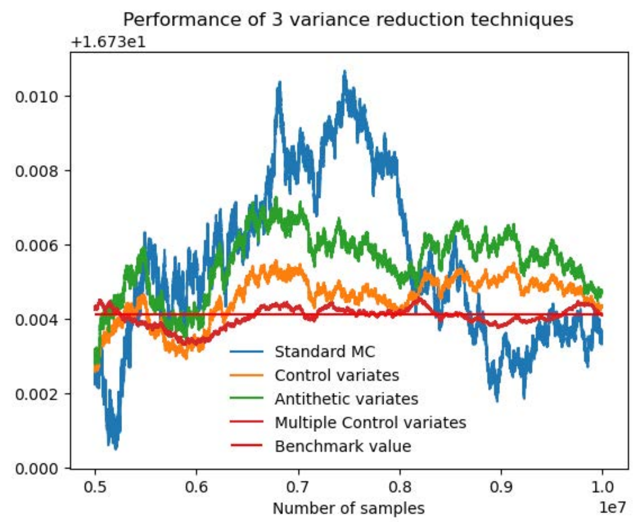


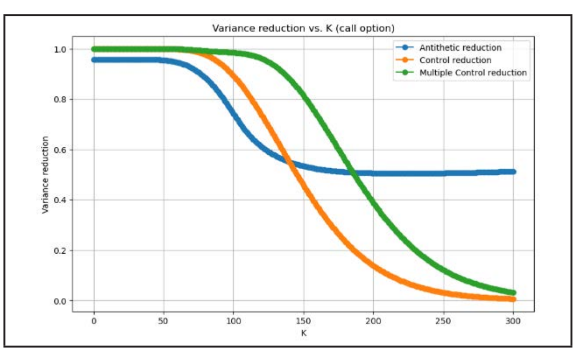
Fig. 1: Comparison of Monte Carlo estimates and Black-Scholes model prices for call option.

This comparison confirms the accuracy of our simulation setup and the effectiveness of the variance reduction techniques.

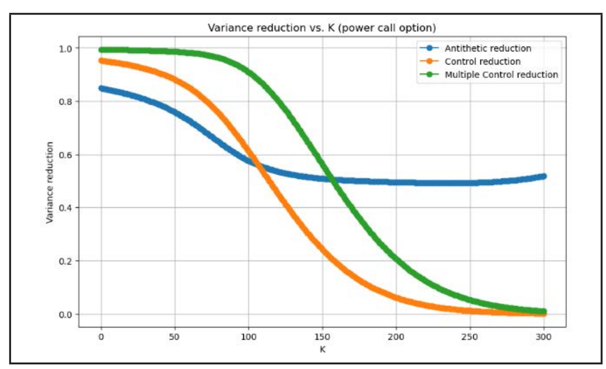
4.2 Impact of Strike Price (K) on Variance Reduction

As shown in the figures (2 and 3). The strike price K showed a significant influences for the effectiveness of variance reduction methods.

4.2.1 Call Options



(a) European Call Option.



(b) Power Call Option.

Fig. 2: Variance reduction for Call Options at different strike prices K.

• For both call option and power call option, there was a Continuous downward trend in the variance reduction for the K values interval between 0 and 300.

• In call option, the effectiveness of reduction variance for multiple control variates method was better the rest of two methods for K under approximately 175. While after that, the antithetic variates method became the optimal.

4.2.2 Put Options

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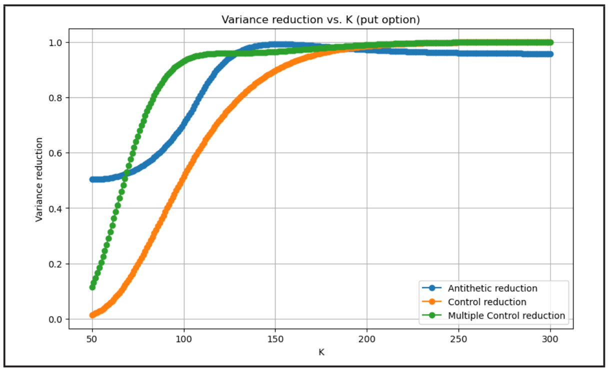
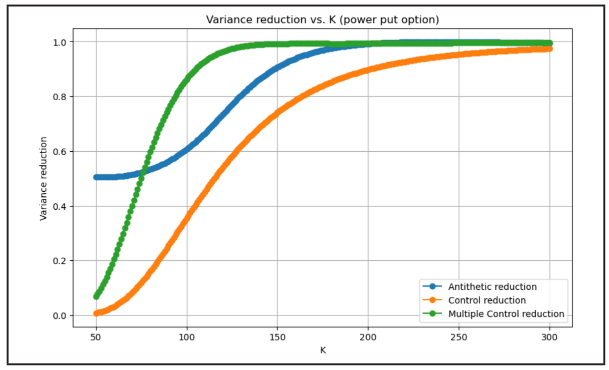


(a) European Put Option.



(b) Power Put Option.

Fig. 3: Variance reduction for Put Options at different strike prices K.

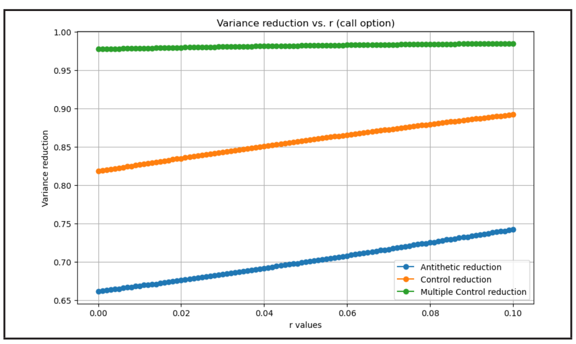
• As to the European put option and power put option, the effect of the variance reduction commonly presented a increase trend.

• In put option, the antithetic reduction reached the peak point for K around 150. After that, it slowly declined. For K around 200, multiple control variates and control variates methods outperform antithetic variates. When K was increasing over 200, the variance reduction of these two methods was basically equal. Similarly, the variance reduction value for the multiple control variates and antithetic variates are nearly the same for K around 200 in power put option.

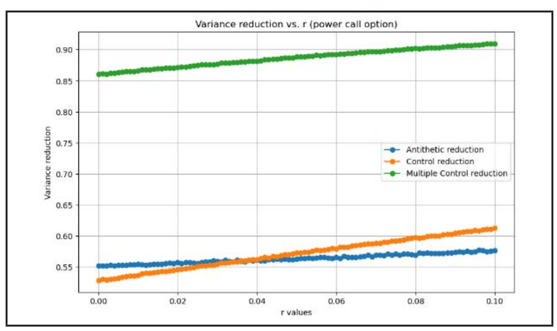
4.3 Impact of Risk-Free Interest Rate (r) on Variance Reduction

As the figures (4 and 5) show, the risk-free interest rate r ranged from 0 to 0.1 and it has a clear impact on the effectiveness of the variance reduction methods. Multiple control variates performed as the best technique in all experimental occasions, whose effect is higher than 95% in European call option and over 85% in other cases. Overall, all techniques performed better for European options compared to power options.

4.3.1 Call Options



(a) European Call Option.



(b) Power Call Option.

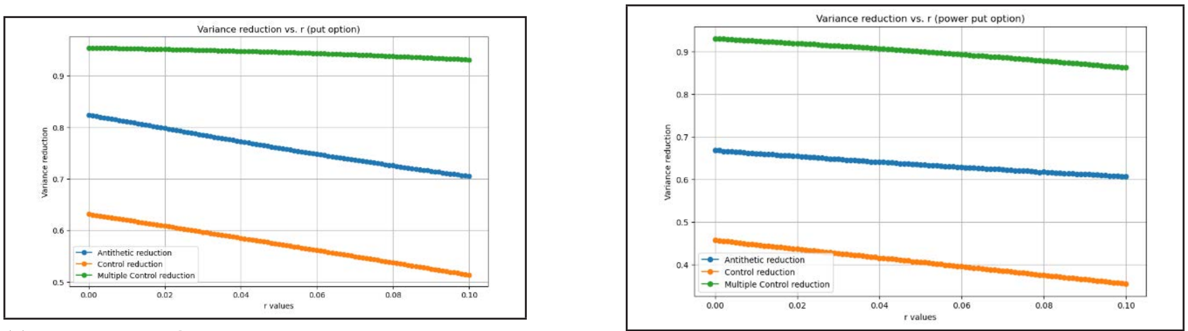
Fig. 4: Variance reduction for Call Options at different risk-free interest rates r.

• For both European and power call options, the variance reduction showed an upward trend as r increased.

• For control variates, the increasing rate was similar in two cases.

• In the case of power options, the performance of control and antithetic variates was quite similar, but control variates exhibited a more rapid increment with increasing r.

4.3.1 Put Options



(a) European Put Option.

(b) Power Put Option.

Fig. 5: Variance reduction for Put Options at different risk-free interest rates r.

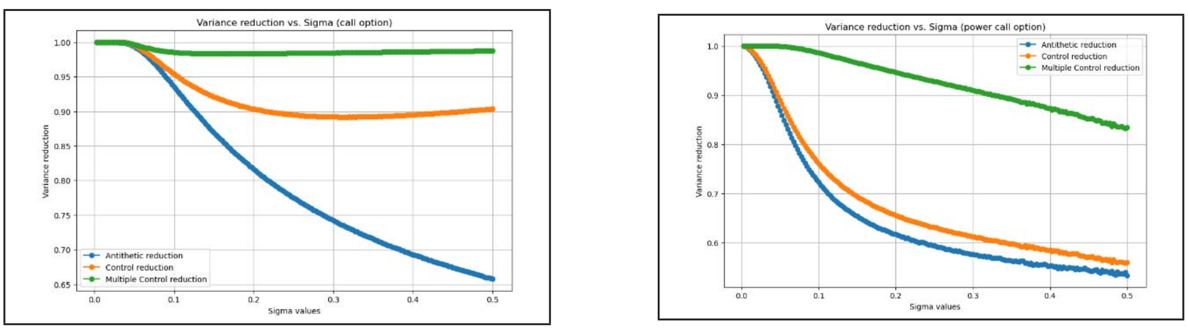
• The variance reduction exhibited a diminishing trend for put options and the effect of antithetic variates was obviously over that of control variates.

4.4 Impact of Volatility (σ) on Variance Reduction

Volatility (σ) ranged from 0 to 0.5 and it was found to have a considerable impact on the variance reduction.

4.4.1 Call Options

Considering 6:



(a) European Call Option.

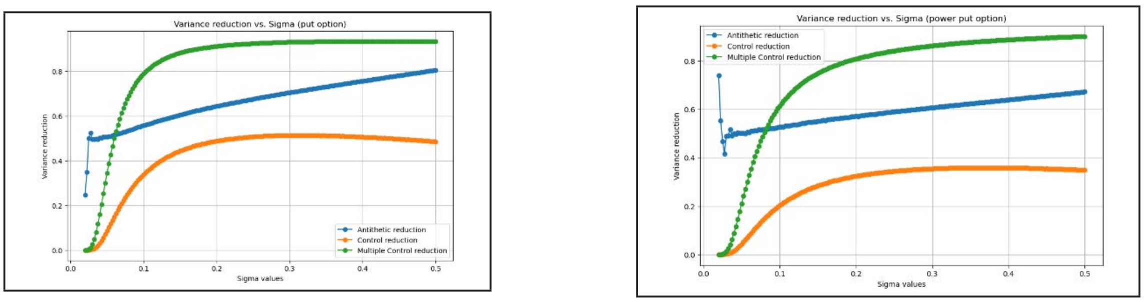
(b) Power Call Option.

Fig. 6: Variance reduction for Call Options at different volatility levels σ.

• For European call options, as σ increased, the effectiveness of the antithetic variates declined sharply, while the control and multiple control variates methods remained relatively stable.

• In power options, the effect of σ was more obvious. The variance reduction decreased sharply for both control and antithetic variates in the range from $\sigma = 0$ to $\sigma = 0.1$. After $\sigma = 0.1$, the figures of them tend to stable while multiple control variates showed a steadily decreasing trend in variance reduction.

4.4.2 Put Options



(a) European Put Option.

(b) Power Put Option.

Fig. 7: Variance reduction for Put Options at different volatility levels σ.

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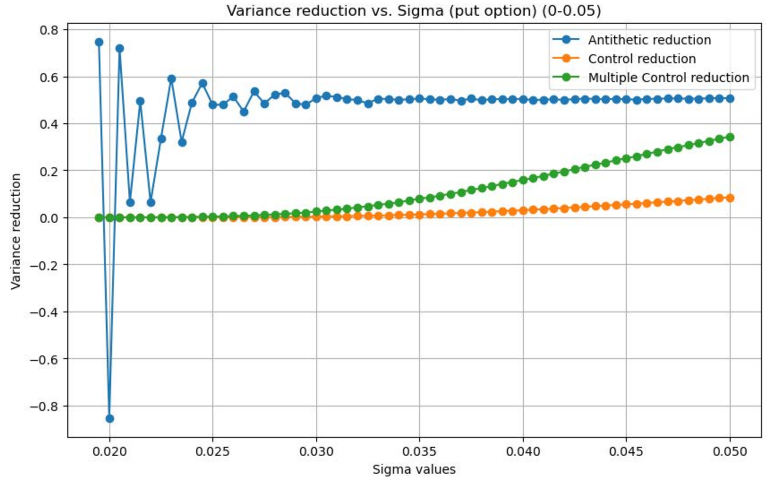

Fig. 8: Effect of low volatility (σ) on the variance reduction techniques for European Put Option.

When sigma is too close to 0, the theory of antithetic variates cannot achieve the effect of reducing variance, as shown in figure (7 and 8). At the same time, we set the minimum value of sigma to 0.03 when analyzing the experimental data because the control variates method cannot have a good effect near 0. After this value, the coding operation of the antithetic method will not have abnormal values.

• For $\sigma < 0.1$, both control and multiple control variates did not perform well but were with rapid increments.

• After σ is larger than 0.1, control and multiple control variates remained relatively stable, as 45% and 90% respectively, while antithetic variates exhibited a steady increase in performance.

5. Conclusion

This research investigated the performance of control variates, multiple control variates, and antithetic variates in reducing the variance of Monte Carlo simulations for option pricing. The research compared how different values of strike price (K), risk-free interest rate (r), and volatility (σ) impacted the effectiveness of these variance reduction techniques.

5.1 Key Findings

• Multiple control variates: In most experimental cases, it is the most efficient technique in variance reduction.

• Strike Price (K): The effectiveness of variance reduction methods is highly dependent on the strike price. For call options, control variates perform well at relatively low strike prices, while antithetic variates excel at higher K values. In cases of put options, antithetic variates performed well in lower K values. Multiple control variates consistently outperformed both in most occasions, especially for put options.

• Risk-Free Interest Rate (r): All variance reduction techniques performed better for European options compared to power options. Multiple control variates showed the best overall performance across different r values, especially for European call options $(>95\%)$.

• Volatility (σ): Volatility had a strong influence on the effectiveness of variance reduction techniques. While control and multiple control variates remained stable across different σ values except for power call option, antithetic variates showed a sharp decline in performance for call options when σ increased. However, in put options, antithetic variates improved steadily as volatility increased. For call options, control variates performed better than antithetic variates while it was less effective compared to antithetic variates for put options.

5.2 Implementation in Practice

The results demonstrate that different variance reduction techniques should be selected based on particular conditions:

• In most cases, such as $(K < 150)$ for call options, $(K > 70)$ for put options and $(\sigma > 0.1)$ for put options, multiple control variates are highly effective. It provides a stable and robust technique.

• For environments with high K in call options, low K in power options and low σ in put options, antithetic variates tend to be the best choice. However, this method might cause computing errors when σ is too close to 0.

5.3 Discussion

This study was limited to European and power options, and future work could further analyse more complex options, even Monte Carlo implementation in other derivatives. Moreover, future research could exploit the implementation of variance reduction techniques in more practical occasions.

5.4 Conclusion

The research shows that the efficiency of Monte Carlo simulations in option pricing could be improved significantly by variance reduction techniques such as control variates, multiple control variates, and antithetic variates. Varying market conditions could result in different choices which give the best variance reduction performance.

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