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Group Theory and the Megaminx

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Abstract:
The Rubik's cube is famous for its complexity, and its applications with the group theory also amaze mathematicians. Based on the Rubik's cube, megaminx is created. Similarly, it can be applied with group theory. This work aims to define the structure of megaminx in terms of the Group Theory and deduce the full conditions for megaminx to stay in a valid configuration of it.

Keywords: Group Theory

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1. Introduction

Erno Rubik invented the popular three dimensional combination puzzle, which is later known as the Rubik's Cube[1]. So far, it has been applied with the Group Theory almost completely by mathematicians; its structure, its valid configurations and its solutions all can be explained in terms of Group Theory. Later, based on the Rubik's Cube, many different puzzles are created, such as pyraminx, megaminx and so on. megaminx is very similar to the Rubik's cube but it is more complicated. Since there is not so much work on megaminx compared to that of the Rubik's cube, this work will focus on applying the Group Theory to the structure of megaminx and deducing the full conditions needed for megaminx to stay in a valid configuration.

2. Basic Concepts

Before constructing the structure and deducing the conditions, a lot of definitions of the group theory are necessary. The necessary definitions in the group theory are listed below, taken mainly from [2] and [3].

Definition 1. *G is a set and we have the opertaion $*$, and we can call G a group under the $*$ operation if the following three properties are satisfied:*

1. If we have elements a, b and c, which are all from G, they should satisfy the following equation: $(a*b)*c = a*(b*c)$, and this is also called associativity.

2. There should be an element e in G which has the property that $b*e = e*b = b$ for all $b \in G$, and e can also be called the identity element.

3. For all the element a in G, b has a correspond b^{-1}, which has the property that $b*b^{-1} = b^{-1}*b = e$, and b^{-1} can be called the inverse.

Definition 2. *A set H of a group $(G,*)$ can be defined as a subgroup of G if we can prove that $(H,*)$ is also a group.*

Definition 3. *If we have a permutation group with n elements, we can use S_n to represent this group and it is called the symmetric group.*

Definition 4. *The cycle $(a_1 a_2 \dots a_k)$ is the element $\sigma \in S_n$ defined by*

$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_k) = a_1$ *and if we have an element $mthatm \notin \{a_1, a_2, a_3, \dots, a_k\}$. The length of this kind of cycle is k. And a cycle of length k can be called a k-cycle, i.e. a cycle of length 2 can be called* *a 2-cycle.*

Definition 5. *Since the permutation can be expressed into the product of several 2-cycle groups, we can use this property to classify them. For those that can be expressed into an even number of 2-cycle groups, we call them even. For those that can be expressed into an odd number of 2-cycle groups, we call them odd.*

3. Megaminx

3.1 Notation

The Rubik's Cube has six faces and they can be defined as U, D, R, L, F, B. Similarly, megaminx has 12 faces, and we can define them as A, B, C, D, E, F, G, H, I, J, K, L, as shown in Figure 1.

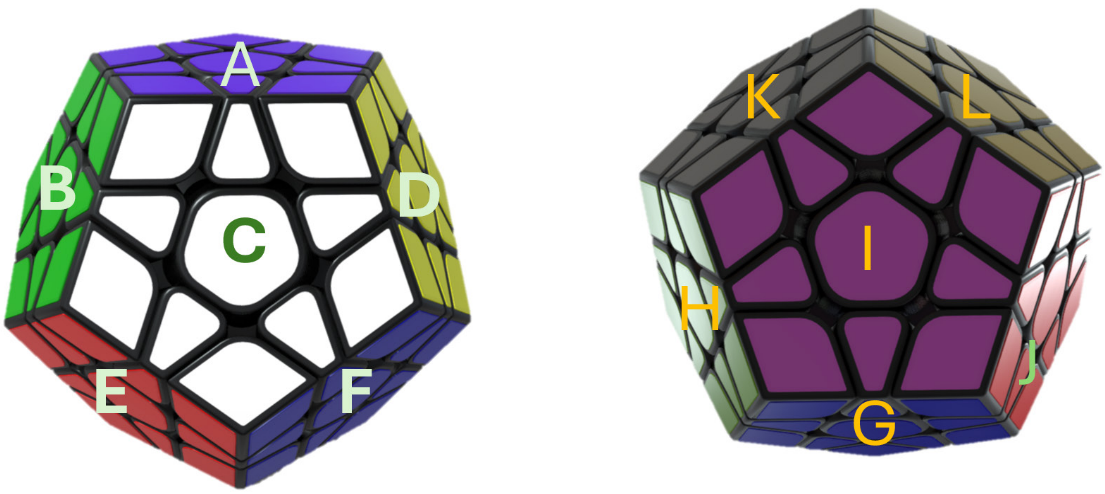

Figure 1 Megaminx model with each face labeled with a letter from A to L

3.2 Cubie Notation

Referring to [4], similar to the Rubik's cube, megaminx only has corner cubies which are the cubies on the corners of megaminx, edge cubies which are the cubies between the corner cubies and on the edges of megaminx, and center cubies which are located at the center of each face and cannot move. There are 20 corner cubies and 30 edge cubies.

Now for each face we have a special label for it, and to name a corner cubie, we can just combine its visible faces in counter-clockwise order. Take the corner cubie on the corner of face A, face D and face C as an example, it can be named as acd. Similarly, for edge cubies, we combine its 2 visible faces, such as ac, bc and so on.

3.3 Movement Notation

Similar to the Rubik's Cube, any configurations of megaminx can be achieved by the combination of the rotations of different faces. Therefore, like the notations we use to describe the movement of the faces of the Rubik's Cube, we use the capital letter to describe the counter-clockwise 72° rotation of the correspond face. For instance, A represents the counter-clockwise 72° rotation of face A. In contrast, we have A^{-1} represent the clockwise 72° rotation of face A.

3.4 Direction Notation

As we define the name for the cubies, we can easily find that for one corner cubie, for example, it has three names, like acd, cda and dac. It has different directions. We can randomly define a face as the corner cubie's top face. For instance, we can define the purple face as abc's top face, and once the purple face is not at the original location and is on the original face C, we call the direction of abc as 2, because it is turned 240°. Similarly, if the purple face is at the original face B, we call the direction of abc as 1. As shown in Figure 2.

Similarly, the direction of an edge cubie can be either 0 or 1.

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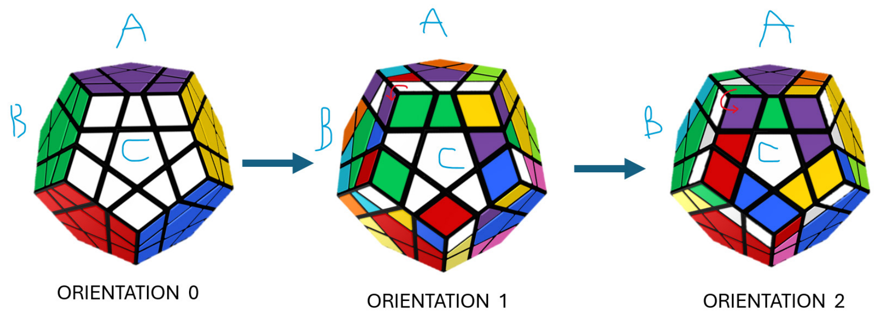

Figure 2 Rotation of abc that changes the direction

3.5 Configuration Notation

To represent the configuration of megaminx, we actually only have to think of four useful data:

1. locations of the corner cubies on megaminx

2. locations of the edge cubies on megaminx

3. directions of the corner cubies on megaminx

4. directions of the edge cubies on megaminx

The locations of the corner cubies can be represented by σ which is from S_{20}, and the locations of the edge cubies can be represented by τ which is from S_{30}.

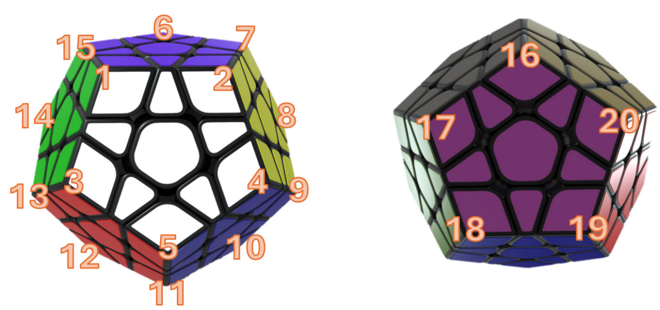

Figure 3 Megaminx model with each corner cubie labeled with a number from 1~20

As shown in Figure 3, we have labeled each corner cubie with a different number. If megaminx is in any configuration, we could describe directions of the corner cubies on megaminx like this: for any i between 1 and 20, we use x_i to represent the direction of one single corner cubie, and we have x for the ordered 20-tuple $(x_1, x_2, x_3, ..., x_{19}, x_{20})$. Since there are 3 possible directions for an corner cubie, we can deduce that x_i as being elements of $\mathbb{Z}_3$. Thus, we can get that $x \in \mathbb{Z}_3^{20}$.

Similarly, we can use y_i to represent the directions of the edge cubies, and $y \in \mathbb{Z}_2^{30}$.

To sum up, the configuration of megaminx could be represented by (σ, τ, x, y).

4. Megaminx Group

Theorem 1. *The megaminx group can be written as $(G,*)$ where G is the set of all possible movements of megaminx, and operation $*$ means that if we write $A*B$, the face A will rotate 72° counter-clockwise and then the face B will rotate 72° counter-clockwise. Besides, the generators of G are*

$G = ?A, B, C, D, E, F, G, H, I, J, K, L?$

Next let's examine whether $(G,*)$ is truly a group or not:

1. Group G is absolutely closed through the operation $*$,

because if P_1 and P_2 are two movements, $P_1 * P_2$ is obviously a movement too.

2. The identity element e_G of group $(G,*)$ is the empty movement since $P * e = P$, there is no change to the movement P.

3. The inverses of the elements are just the clockwise rotations of the faces, so group $(G,*)$ obviously has inverses.

4. If we use C to represent a turned cubie, we then use $P(C)$ to show the turned cubicle where the cubie C ends up in after we operate the movement P. Therefore we can get that

$$(P_1 * P_2)(C) = P_2(P_1(C))$$

Similarly, we have

$$((P_1 * P_2) * P_3)(C) = P_3((P_2 * P_1(C)) = P_3(P_2(P_1(C)))$$

$$(P_1 * (P_2 * P_3))(C) = (P_2 * P_3)(P_1(C)) = P_3(P_2(P_1(C)))$$

Therefore, we can get that $(P_1 * P_2) * P_3 = P_1 * (P_2 * P_3)$, which means $*$ is associative.

However, we need to be cautious that although $*$ is associative, it is not abelian. Using the same method, we can prove this.

$$(P_1 * P_2)(C) = P_2(P_1(C))$$

$$(P_2 * P_1)(C) = P_1(P_2(C))$$

Therefore, we can get that $P_1 * P_2 \neq P_2 * P_1$, which means $*$ is not abelian.

To sum up, megaminx group $(G,*)$ is a group.

5. Megaminx valid configurations

We use (σ, τ, x, y) to represent the configuration of the megaminx, where σ indicates the locations of the corner cubies on the megaminx, τ reveals the locations of the edge cubies on the megaminx, x shows the directions of the corner cubies on the megaminx and y suggests the directions of the edge cubies on the megaminx.

Theorem 2. *A configuration of megaminx (σ, τ, x, y) is valid only if it satisfies the following conditions:*

1. $sgn\,\sigma = sgn\,\tau = 1$

2. $\sum x_i \equiv 0$ *(mod3)*

3. $\sum y_i \equiv 0$ *(mod2)*

The rest of the paper will projet the proof of this theorem, and we get some idea for the proof from [4] .

First, let's prove that under the valid configuration (σ, τ, x, y), we have the 3 conditions above.

If we have a scrambled megaminx, it is definitely scrambled through the combination of the rotations of different faces from an original megaminx. Thus, we have

$$g = P_1 P_2 P_3 \ldots P_k$$

$$P_i \in \{A, B, C, D, E, F, G, H, I, J, K, L\} \text{ and } g \in G$$

It is easy to prove that

$$\forall P \in \{A, B, C, D, E, F, G, H, I, J, K, L\},$$

$$sgn(\sigma(P)) = sgn(\tau(P))$$

Just take operation C as an example. After C, as shown in Figure 3, $\sigma = (1,3,5,4,2) = (1,2)(1,4)(1,5)(1,3)$ and so $sgn\,\sigma = 1$, and $\tau = (1,5,4,3,2) = (1,2)(1,3)(1,4)(1,5)$ and so $sgn\,\tau = 1$, $sgn\,\sigma = sgn\,\tau = 1$.

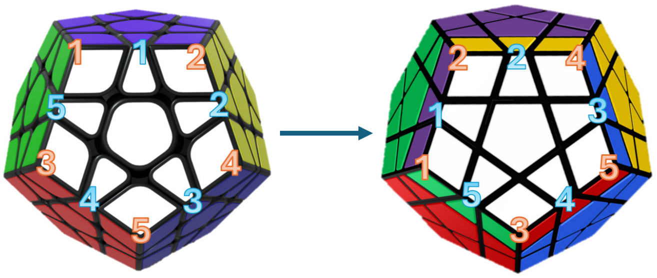

Figure 4 Megaminx model after operation C

Since we know that

$$sgn(\sigma\rho\tau) = sgn(\sigma)sgn(\rho)sgn(\tau)$$

As a result, we can deduce that

$$sgn(\sigma(g)) = \prod_{i=1}^{k} sgn(\sigma(P_i)) = \prod_{i=1}^{k} sgn(\tau(P_i)) = sgn(\tau(g))$$

Thus, we have proved that if the configuration (σ, τ, x, y) is valid, $sgn\,\sigma = sgn\,\tau$.

For the next two conditions, first we need to define the top face of the cubies. We define the white or grey faces as the top face of those corner cubies that have white or grey face, and for those that do not have white or grey face, we define the purple or pink faces as the top face of them, and

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for those that do not have white or grey or purple or pink face, we define the yellow or banana face as their top face. First, let's think, if we just do operation C or I, the direction of the corner cubes will actually have no change since their top faces did not change a bit. And the sum is obviously can be divided by 3 since it is 0.

If we think of other operations, $A, B, D, E, F, G, H, J, K, L$, and we can see that the changes are symmetrical and actually under those operations, a corner cubie will be scrambled down off face *C* and the other corner cubie will be scrambled up onto face *C*. Therefore, we can deduce that the elements of x are either fell by *1(mod3)* or rose by *1(mod3)* respectively. Which means that, as they are added together, *1(mod3)* and *-1(mod3)* would balance each other and finally get 0. Which means that finally, the sum of the directions would be 0 and it is obviously divided by 3. So the second condition is also proved. And think of the same method, the third condition is proved.

Therefore, we have proved that if the configuration (σ, τ, x, y) is valid, the three conditions exist, which means we have proved one direction of theorem 3.3.1. Next, we will prove the other direction.

Assume that $sgn\,\sigma = sgn\,\tau$, $\sum x_i \equiv 0$ *(mod3)* and $\sum y_i \equiv 0$ *(mod2)*, we have

1. If (σ, τ, x, y) is a configuration of megaminx such that $sgn\sigma = sgn\tau, \sum x_i \equiv 0(mod\,3)$ and $\sum y_i \equiv 0(mod\,2)$, then there must be a movement $P \in G$ so that $(\sigma, \tau, x, y) \cdot P = (1, \tau', x', y')$ with $sgn\tau' = 1, \sum x_i' \equiv 0(mod\,3)$ and $\sum y_i' \equiv 0$ *(mod2)*, which means that the locations of the corner cubies are right.

2. If $(1, \tau, x, y)$ is a configuration of megaminx such that $sgn\tau = 1, \sum x_i \equiv 0(mod\,3)$ and $\sum y_i \equiv 0(mod\,2)$, then there must be a movement $P \in G$ so that $(1, \tau, x, y) \cdot P = (1, \tau', 0, y')$ with $sgn\tau' = 1$ and $\sum y_i' \equiv 0(mod\,2)$, which means that the directions of the corner cubies are right.

3. If $(1, \tau, 0, y)$ is a configuration of megaminx such that $sgn\tau = 1 and \sum y_i \equiv 0(mod\,2)$, then there must be a movement $P \in G$ so that $(1, \tau, 0, y) \cdot P = (1, 1, 0, y')$ with $\sum y_i' \equiv 0(mod\,2)$, which means that the locations of the edge cubies are right.

4. If $(1, 1, 0, y)$ is a configuration of megaminx such that $\sum y_i \equiv 0(mod\,2)$, then there must be a movement $P \in G$ so that $(1, 1, 0, y) \cdot P = (1, 1, 0, 0)$, which means that the directions of the edge cubies are right, and the cube can be solved.

To sum up, we have proved both the two directions of theorem 2.

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