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The Investigation of 15-puzzle, Varikon Boxes, and Similar Variants

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Abstract:
The Varikon box, a 3-dimensional variant of the 15-puzzle, is the focus of this investigation. The goal is to investigate three questions: Whether swapping positions of two of its blocks affect its solvability, the number of distinct configurations of the puzzle, and the least number of steps needed for solving the 15-puzzle. First, we introduced our investigation with the same questions, yet for 15-puzzle, specifically using permutation parity to testify one's solvability, then finding the pattern vice versa that is possible 15-puzzle configurations, and then all of it is mirrored onto the study of Varikon Boxes and other permutational puzzles.

Keywords: The Varikon box; 15-puzzle; permutational puzzles; permutation parity

1 The Basics

1.1 Groups and Permutation

Before all investigations begin, the basics of groups and permutations should be listed:

Definition 1. *A group* G *is a set* S *with an operation:* S ⋆ S → S, *where*

- *the binary operation* (⋆) *is associative*
- *identity* $e \in$ G
- $\forall$ *element* $a \in$ G, $\exists b \in$ G : $a \star b = b \star a = e$

Definition 2. *A* permutation σ *of a set* S *is a bijection such that* σ: S → S. (Chapple et al., 2000)

Notation:

$$\sigma = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ b_1 & b_2 & \dots & b_{n-1} & b_n \end{pmatrix}, a_i, b_i \in S, n \in \mathbb{Z}^+$$

Definition 3. *The inverse of permutation*

$$\sigma = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ \sigma(a_1) & \sigma(a_2) & \dots & \sigma(a_{n-1}) & \sigma(a_n) \end{pmatrix}, a_i \in S, n \in \mathbb{Z}^+ :$$

is:

$$\sigma^{-1} = \begin{pmatrix} \sigma(a_1) & \sigma(a_2) & \dots & \sigma(a_{n-1}) & \sigma(a_n) \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}, a_i \in \boldsymbol{S}, n \in \mathbb{Z}^+$$

Definition 4. *Any permutation in the form of the fol-*

lowing:

$$\sigma = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_2 & a_3 & \dots & a_n & a_1 \end{pmatrix}, a_i \in S, n \in \mathbb{Z}^+$$

could be expressed as a cycle*:* $(a_1 a_2 \dots a_{n-1} a_n)$.

Definition 5. *Any permutation could be expressed as the product of transpositions (2-cycles):*

$$\sigma = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_2 & a_3 & \dots & a_n & a_1 \end{pmatrix}, a_i \in S, n \in \mathbb{Z}^+$$

could be expressed as: $(a_1 a_2)(a_2 a_3)\dots(a_{n-2} a_{n-1})(a_{n-1} a_n)$.

(Chapple et al., 2000)

2 The 15-puzzle

2.1 Introduction

Before heading straight into the investigation of the Varikon Box, an easier concept to start with these sorts of puzzles is the 15-puzzle, which is the most fundamental form of permutation puzzles. The game is formed by a 4 by 4 grid, whereas 15 of those grids are numbered with the integers from 1 to 15, and the remaining block is empty to enable the sliding of different pieces. The most basic rule of this game is to unscramble the 15-puzzle into the following arrangement:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Fig 2.1.1 The goal of 15-puzzle

There are two important definitions needed for further mathematical operations:

Definition 6. *A* configuration *of the 15-puzzle is any arrangement of the 15 number blocks of which the 16*th *block is empty.* (Powell *et al.*, 2020)

Definition 7. *The* possible configuration *of the 15-puzzle is any configuration of the 15-puzzle that could be formed by normal procedures (sliding blocks, but not swapping them).*

From the two definitions above, it is easy to understand that any possible configuration P ∈ S_{15}, whereas S_{15} is the set of the 15 numbers. Furthermore, since every configuration corresponds to a specific grid (one-to-one), and is an onto function, hence P is a permutation, and more specifically, a cycle.

Three questions are formed from this game: Can this puzzle be solved when any two numbered blocks exchanged their positions with each other? How many distinct possibilities of configurations are there for this puzzle? At least how many steps should be taken to ensure the puzzle is solved?

2.2 Switching blocks

One of the most important focuses of the 15-puzzle is whether switching positions of any two blocks would affect the solvability of this puzzle.

The 15-puzzle, as explained in previous paragraphs, would always be a permutation P of which P □ S_{15}. Accordingly, the question now becomes whether the permutation P is possible to become the permutation formed by the two swapped numbers. In other words:

$$\forall a_1, a_2 \in \{1, 2, \dots, 15\}(a_1 \neq a_2), \exists P : P = (a_1 a_2)$$

To be able to prove that it is impossible for a 15-puzzle with two swapped numbers to be solved, the characteristic of the parity of permutations could be considered.

It is first crucial to realize that all of the permutations possible to happen in a 15-puzzle are even.

Taking the bottom right corner of the puzzle as an example:

11	12
15	

Fig 2.2.1 The bottom right corner, unscrambled

The Figure above (Fig 2.2.1) is the bottom right corner of the 15-puzzle in its unscrambled form.

Now, the question is, what is the possible configuration of this corner, if only these three number blocks consist of sliding movement?

11	12
	15

	12
11	15

12	
11	15

12	15
11	

Fig 2.2.2 The bottom right corner, configuration-1

a) "15" move rightwards [top-left]
b) "11" move downwards [top-middle]
c) "12" move leftwards [top-right]
d) "15" move upwards [bottom]

As shown in Figure (Fig 2.2.2 d)), this is one of the configurations of the bottom-right corner of the 15-puzzle. Expressing in cycles, this permutation is: σ = (11 12 15). Clearly, σ is an even permutation, since σ could be

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expressed as σ = (11 12) (12 15), the product of an even number of transpositions. (Conrad, n.d.) Now, the job became proving all permutations in a 15-puzzle an even permutation. To be able to do that, it is necessary to find another direction to analyse σ. Let the empty block be labelled "0", then the moves done in Fig 1.2.2 would become σ = (15 0) (11 0) (12 0) (15 0), which is also an even number of transpositions. What's more important is that now the relationships of the number of steps in different orientations could attend to the determination of the parity of the permutations.

This could be done by quantifying the moves into numbers, which is called the sliding index (α) in this investigation. Defining the sliding index of a slide rightwards as $+\hat{x}$, and a slide upwards as $+\hat{y}$, (meaning a slide leftwards is represented by $-\hat{x}$, and a slide downwards is $-\hat{y}$.) As the unscrambled state of the 15-puzzle being defined as α = 0, every possible configuration of the 15-puzzle should also satisfy α = 0. Below is an example:

1	2	3	4
5	6	7	8
9	10	11	12
13	14		15

1	2	3	4
5	6	7	8
9	10	11	12
13		14	15

1	2	3	4
5	6	7	8
9		11	12
13	10	14	15

1	2	3	4
5	6	7	8
9	11		12
13	10	14	15

1	2	3	4
5	6	7	8
9	11	12	
13	10	14	15

1	2	3	4
5	6	7	8
9	11	12	15
13	10	14	

Fig 2.2.3 One of the possible configurations

(Steps arranged left-to-right)

The permutation shown above could be written as: σ = (10 11 12 15 14). In the form of sliding indexes, it is expressed $\alpha = 0 + (+\hat{x}_{[15]}) + (+\hat{x}_{[14]}) + (-\hat{y}_{[10]}) + (-\hat{x}_{[11]}) + (-\hat{x}_{[12]}) + (+\hat{y}_{[15]}) = 0$.

For movements accounting for horizontal directions, since only $-\hat{x}$ could cancel out $+\hat{x}$ ($+\hat{x}$ + $-\hat{x}$ = 0), it is obvious that *the number of slides leftwards is equal to the number of slides rightwards*; similarly, since only $-\hat{y}$ could cancel out $+\hat{y}$, *the number of slides upwards is equal to the number of slides downwards*.

From the theories above, it is obvious that the sum of the number of steps of any possible configuration would be an even number. Additionally, each movement could be expressed mathematically as a transposition of (*a* 0), $a \in \{1, 2, \ldots, 15\}$, hence *every configuration could be expressed as a product of an even number of transpositions*, indicating that *every possible configuration of the 15-puzzle is an even permutation*.

Lemma 1. *Every possible configuration of the 15-puzzle is an even permutation in* S_{15}

The other way to prove this theorem is by looking at the movement of the empty block. Still using the sliding index as a scale for the movement of the empty block, (movement rightwards = $+\hat{x}$, leftwards = $-\hat{x}$, upwards = $+\hat{y}$, downwards = $-\hat{y}$) the fact that the permutation of all possible 15-puzzle configurations is even could be proved by the fact that for each permutation, the empty block must return to the 16th slot. (Williams, 2020) In other words, the *displacement* of the empty slot for each configuration is 0.

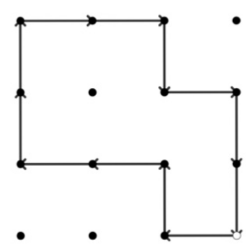

Fig 2.2.4 Example of empty slot movement

It is obvious that to get to a displacement of 0, the number of steps rightwards should be equivalent to the number of steps leftwards, the same relationship should be held between upwards and downward movements.

Nevertheless, a swap of two number blocks is a single permutation, since (*a1 a2*), *a1*, *a2* ∈ {1, 2, …, 15} (*a1* ≠ *a2*) is a single permutation. Therefore, what is left to prove is that a permutation is either even or odd, but not both.

But before that, we should prove that the identity is an even permutation, but not odd.

Lemma 2. *Every permutation in* S_n *(n > 1) is either even or odd, but not both.*

Proof:

$$Let M_n = \prod_{1 \leqslant i \leqslant j \leqslant n} (x_i - x_j), n \in Z^+$$

$$Thus, M_2 = \prod_{1 \leqslant i \leqslant j \leqslant 2} (x_i - x_j) = x_1 - x_2$$

$$M_3 = \prod_{1 \leqslant i \leqslant j \leqslant 3} (x_i - x_j) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3), etc.$$

let permutation σ = S_n, and *a* = the number of factors, then

$$\sigma \cdot M_n = (-1)^a M_n = sig(\sigma) \cdot M_n$$

∴ if σ is even and odd at the same time, then $M_n = -M_n$

but, $\because\ M_n \neq -M_n$
$\therefore$ permutation σ is either even or odd, but not both
Q. E. D.
Now, since swapping two numbers is an odd permutation, and every possible configuration of the 15-puzzle is an even permutation (Lemma 1.), as even permutations and odd permutations are non-interchangeable (Lemma 2.), it is obvious to understand that,
Theorem 1. *It is impossible to solve a 15-puzzle with two of the numbers swapped.*
Then, by pattern, it is to check if having two empty slots would solve the unsolvable.
Since the two empty slots are completely identical, hence their exchange would not do anything to the permutation, but it will change the math, since exchanging these two empty slots is an odd permutation (0 0), whereas apparently, it does not change, and hence in this "14-puzzle" (where there are 2 empty slots), the odd permutation and even permutation could be transferred into each other, hence becoming a solvable puzzle, hence:
Theorem 2. *All configurations of a 14-puzzle (two empty slots) are solvable.*

2.3 Possible configurations

Now, it is already known from Lemma 1 that every possible configuration of the 15-puzzle is an even permutation, but this leads to interesting investigations of the authenticity of the converse of this theorem, which is the statement "Every alternating group in S15 of the 15-puzzle could be solved."
Since it is an obvious fact that all 3-cycles in S15 of the 15-puzzle is solvable, hence this problem becomes:
Lemma 3. *Every even permutation could be written as the product of 3-cycles.*
Proof:
let even permutation $\sigma=(a_1a_2\ldots a_n), n\in\mathbb{Z}^+, 2\nmid n$
$\therefore \sigma=(a_1a_2)(a_2a_3)(a_3a_4)\ldots(a_{n-2}a_{n-1})(a_{n-1}a_n)$
$=(a_1a_2a_3)(a_3a_4a_5)\ldots(a_{n-2}a_{n-1}a_n)$
$\therefore$ even permutation σ could be written as the product of 3-cycles.
Q.E.D.
This lemma would help for the proof of the following theorem:
Theorem 3. *Every alternating group in* S_{15} *is a solvable permutation in the 15-puzzle.*
Proof:
$\because$ every 3-cycle in S_{15} is solvable
every even permutation could be written as the product of 3-cycles
$\therefore$ every even permutation (alternating group) in S_{15} is a solvable permutation in the 15-puzzle.
Q.E.D.
Theorem 4. Every configuration (including the possible and impossible) of the 15-puzzle could be unscrambled or a swap of (14, 15) away from its unscrambled state.
Proof:
if $sgn(\sigma)=1$, configuration could be unscrambled (proved in Theorem 2.)
if $sgn(\sigma)=-1$, let $\tau=(1415)\cdot(\sigma\cdot(1415))$
$sgn(\sigma\cdot(1415))=1?canbeunscrambled=indentitye$
$\therefore \tau=(1415)\cdot e=(1415)$
Q.E.D.
Now, this solvability could also be found within the type of so-called "coiled 15-puzzle":

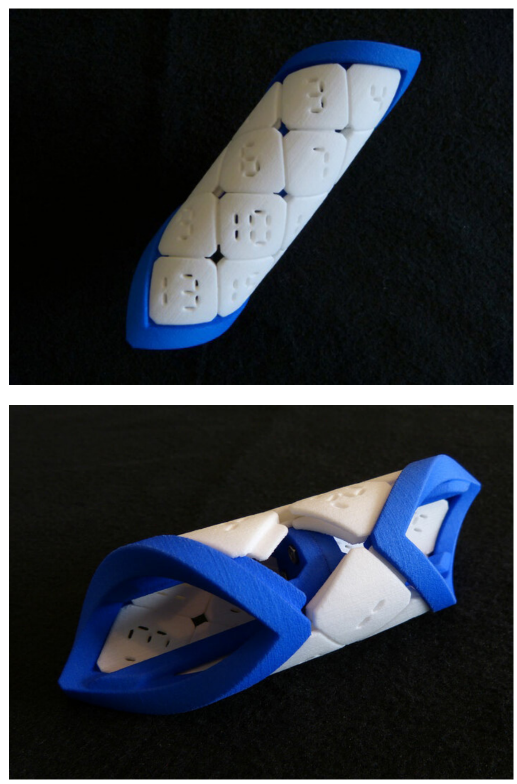

Fig 2.2.5 Coiled 15-puzzle

For this puzzle, it is possible for blocks at top-left corners to move into the bottom-right corner of the puzzle, meaning under the unscrambled state, an additional (1 0) is allowed. Since (1 0) is an odd permutation, hence the transference between odd and even permutations is possible in this puzzle, meaning that again:
Theorem 5. *All configurations of a coiled 15-puzzle are solvable.*
After proving that all possible configurations of the 15-puzzle are even permutations, the following step is to calculate the number of possible permutations of the

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15-puzzle.
To create convenience for calculation, two symbols are being defined as the following:
G_{15} = group of all the possible configurations in the 15-puzzle
A_{15} = alternating group in B_{15}
This leads to a problem:
Problem 1. How many possible configurations are there for 15-puzzle?
Solution:

$\because$ According to Lemma 1, $G_{15} \leqslant A_{15}$

According to Theorem 5, $A_{15} \leqslant G_{15}$

$\therefore G_{15} = A_{15}$

$$\therefore |G_{15}| = |A_{15}| = \frac{15!}{2} = 653837184000$$

3 Varikon Box

3.1 Introduction

Similar to the 15-puzzle, the Varikon Boxes are a type of permutation game, literally the three dimensional variants of the 15-puzzle. One of the simplest Varikon Boxes is a 2 by 2 by 2 space, where 7 solid blocks and an empty slot (for space for sliding) make up its structure. Rather than being labelled by numbers, today's most of the Varikon Boxes are labelled with different colours (as shown in Fig 3.1.1), whereas when unscrambled, its outer faces being monochromatic (shown blue), and the inner faces of the cubicles having another colour (shown red). (D'eon and Nehaniv, 2020)

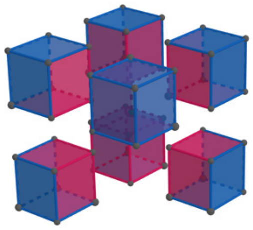

Fig 3.1.1 Dissection of an Unscrambled 2 × 2 × 2 Varikon box

For Varikon boxes, the calculation might be hard, especially there are only colors. However, since the cubicles are not allowed to rotate, it is still possible for people to label them in numbers:

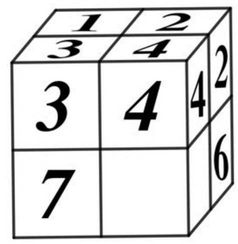

Fig 3.1.2 A numerically labelled 2 × 2 × 2 Varikon box

3.2 Permutation of Varikon Boxes

If, again, using P as the permutation of any possible configuration of the 2 × 2 × 2 Varikon box, then the question of whether swapping numbers would affect the solvability of this puzzle would become:

$\forall a_1, a_2 \in \{1, 2, \ldots, 7\} (a_1 \neq a_2), \exists P : P = (a_1 a_2)$

But, like the similar proof for the 15-puzzle, that this should also be proofed based on several lemmas.
The first job to do is to prove the following:
Lemma 4. *Every possible configuration of a Varikon box is an even permutation*
This could be done using a similar method proving the 15-puzzle even, which is to use the characteristic that the displacement of the empty slot is 0. (As shown in Fig 3.1.2)

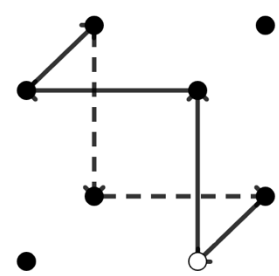

Fig 3.1.3 Example of empty slot movement in Varikon box

Since the displacement of the movement of the empty slot is 0, hence the displacement of its movement on all the three dimensions of the box is also 0, of which meaning that for each movement upwards, there is a movement downwards, for each movement leftwards, there is a corresponding rightwards movement, this rule stays the same for frontwards and backwards movements.
This correspondence, resembling the 15-puzzle, appears in pairs, of which could prove that every possible permutation of the Varikon box is, again, an even permutation.
Since Lemma 4 is proved, it is now possible to prove Theorem 4.
Theorem 6. *It is impossible to solve a 2× 2 × 2 Varikon*

Box with two of the numbers swapped.
Proof:
∵ According to Lemma 4, $sgn(P)=1$
According to Lemma 2, a permutation cannot be both even and odd
$(a_1a_2), a_1, a_2 \in \{1,2,\ldots,7\}(a_1 \neq a_2)$ is odd
∴Impossible
$\nexists P: P=(a_1a_2), a_1, a_2 \in \{1,2,\ldots,7\}(a_1 \neq a_2)$
Q.E.D.

4 a^n-1 Puzzles

This is the chapter of combining all the results together into a a^n*-1* puzzle.
The *a* here means the size of the puzzle (the side length of the square, cube, hypercube…)
Whereas the *n* here meaning the dimension of the puzzle.
Like all the other puzzles this paper has investigated, the processes are analysed in reverse order.
For all the puzzles in previous chapters (15-puzzle and the Varikon box) the pattern of its possible configurations being an even permutation is found, so maybe a curious check of this pattern on other similar permutational puzzles is plausible:
Theorem 7. *Every possible configuration of an* a^n*-1 puzzle is an even permutation*
Again, for all the possible permutations of these puzzles, the empty square needed to return to its position of its unscrambled state, of which requires a displacement of 0.

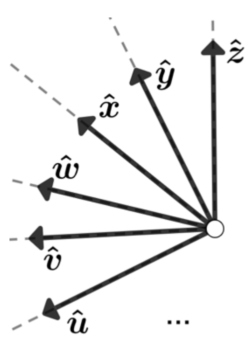

Fig 4.1 Multidimensional axes

Since the displacement is 0, the displacement on *x*-axis is 0, on *y*-axis also 0, so as the *z*-axis, *w*-axis, etc. This indicates that for every unit movement of the empty slot on either axis, there is a correspondingly opposite unit movement that cancels this effect, ensuring that the displacement is 0. This leads the count of steps (transpositions) an even number.
Hence, they are all even permutations.
Accordingly, the next Theorem:
Theorem 8. *It is impossible to solve any* a^n*-1 puzzle with two of the blocks swapped*
Proof:
∵ According to Lemma 4, $sgn(P)=1$
According to Lemma 2, a permutation cannot be both even and odd
$(m_1m_2), m_1, m_2 \in \{1,2,\ldots,a^n-1\}(m_1 \neq m_2)$ is odd
∴Impossible
$\nexists P: P=(m_1m_2), m_1, m_2 \in \{1,2,\ldots,a^n-1\}(m_1 \neq m_2)$
Q.E.D.

5 Algorithms to Solve (n^2-1)-Puzzles

Theorem 9. *Finding the solution that takes the least step of moves for a (n^2-1)-puzzle is NP-hard.*
Erik D. Demaine and Mikhail Rudoy offered a simple prove for this. They proved that the rectilinear Steiner Tree problem, which is an NP-hard problem, can be converted into a (n^2-1)-problem, and the conversion can be done in polynomial time.
The rectilinear Steiner tree problem is about finding a tree that passes through all the given points in a given plane. The tree's total length needs to be no greater than a certain value, and its edges are all rectilinear.

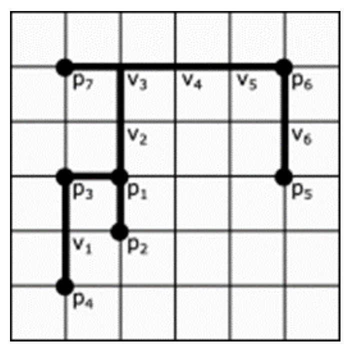

Fig 5.1 an example of a tree connecting points P1, P2, P3, P4, P5, P6, P7

The main idea of Erik D. Demaine and Mikhail Rudoy is the following:
For a given (n^2-1)-puzzle, if a person can always determine whether it is possible to move from a puzzle configuration s to another puzzle configuration t in no more than k steps, then the optimal solution can be found, because the least k can be found. In order to move from puzzle configuration s to puzzle configuration t, a series of permutations of three-cycles need to be constructed. A permutation of three-cycle can be achieved only when the empty block is moved beside the three blocks that need to permute. For a given pair of s and t, all the places that three-cycle permutations need to take place can be determined. The key to find the optimal solution is how to

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move the empty block to all the places that need to have three-cycle permutations in a route that takes the least steps.

For each edge in the route, the empty block needs to pass the edge twice (once in each direction). This is because the second time the empty block passes some path, it needs to undo the effect of the first pass, so that only the effects of three-cycle permutations can stay. So, the moves of the empty block need to be on a tree, and whether it is possible to construct such a tree with some limited total k can be converted to the rectilinear Steiner Tree problem.

Since it is NP-hard to find the optimal solution, we need to search for algorithms that are not always the optimal but can solve the problem in polynomial time.

Algorithm 1. Greedy Search.

Find the block with the number 1 on it. There is always a sequence of moves to take it to the top left corner, although it may disturb other blocks along the way. Similarly, the block with number 2 on it can also be moved to its original position. On the way to move block number 2, it may disturb other blocks, but there will always be a way to move it without disrupting block number 1, because block number 1 is already in its solved position. For the blocks that belong to the first row, each time the algorithm focuses on sending only one block to its original position while keeping the blocks that are already solved on the first row undisturbed. In order to give space to let the last block of the first row move in, some earlier solved blocks may need to be disrupted temporarily, but they can be moved back easily and the first row can be completely solved. After this, the whole first row will not be disrupted again. Similarly, the algorithm can solve the first column. Once the first row and first column are in place, the remaining is an $((n-1)^2-1)$-puzzle, and it can be solved recursively. A benefit of this algorithm is that humans can easily use it in casual games.

According to Ian Parberry, it takes $\frac{8}{3}n^3$ expected moves to solve an (n^2-1)-puzzle, and the worst case for the 15-puzzle is 80 moves.

Algorithm 2. IDA-Star.

Define H as a heuristic estimation of how many steps are needed in a certain situation. Define G as the total number of steps already taken from the starting point. Define B as the bound, which is the guess of the total number of steps needed. This algorithm records G and calculates a new H each time when a move is being taken. Consider the different moves as in a structure of branches consist of "parent" and "children", each move is the "parent" move of its "child" moves. The algorithm explores different moves and the "child" moves of the moves. It will not stop exploring a branch of moves unless the branch is considered to be blocked. A branch is blocked if the parent move makes the sum of G and H bigger than the bound, or if all of its child moves make the sum of G and H bigger than the bound. If all the possible branches are blocked, then it will increase the bound to the smallest G+H found previously, and explore the branches that have the smallest G+H again. It repeats this process of exploring until it gets to the solved.

The heuristic estimation H is the core of the algorithm, and there are various types of ways to calculate H. H needs to be less than or equal to the real number of steps needed, so that the algorithm will not miss the chance to explore the best branch because its H is too big. One possible way is to find the sum of the Manhattan distance of each block from its original position, and then add the number of linear conflicts multiplied by 2. Manhattan distance can be understood as vertical distance plus horizontal distance, which corresponds to least number of moves. A linear conflict happens when two blocks are in the same row or same column, and their original positions are also in the same row or column, and they have to go pass each other to return to their original positions. So at least two extra moves are needed to solve a linear conflict.

This algorithm can be more efficient when the side length of the puzzle, which is n, is very large.

References

D'eon, J. and Nehaniv, C. (2020). ALGEBRAIC STRUCTURE OF THE VARIKON BOX. [online] Available at: https://arxiv.org/pdf/2006.01296v1.

Conrad, K. (n.d.). THE 15-PUZZLE (AND RUBIK'S CUBE). [online] Available at: https://kconrad.math.uconn.edu/blurbs/grouptheory/15puzzle.pdf.

Chapple, A., Croeze, A., Lazo, M. and Merrill, H. (2000). AN ANALYSIS OF THE 15-PUZZLE. [online] Available at: https://www.researchgate.net/publication/265231296_AN_ANALYSIS_OF_THE_15-PUZZLE [Accessed 21 Aug. 2024].

Powell, C. *et al.* (2020) *A mathematical analysis of the 15 puzzle*. https://calebmpowell.com/files/15_Puzzle.pdf.

Williams, T. (2020). Parity in practice: The 15-Puzzle. [online] Groups Made Simple. Available at: https://groupsmadesimple.wordpress.com/2020/06/06/parity-in-practice-the-15-puzzle/ [Accessed 22 Aug. 2024].

Demaine, E. and Rudoy, M. (2018) *A simple proof that the $(n^2 - 1)$-puzzle is hard.* [online] Available at: https://www.sciencedirect.com/science/article/pii/S0304397518302652 [Accessed 22 Aug. 2024]

Parberry, I. (2015) *Solving the (n2 − 1)-Puzzle with 8/3 n3 Expected Moves* [online] Available at: https://www.researchgate.net/publication/281978970_Solving_the_n2_-_1-Puzzle_

with_83_n3_Expected_Moves [Accessed 25 Aug. 2024]
Contributors of Algorithms Insight. (2018) Implementing A-star(A*) to solve N-Puzzle [online] Available at: https://algorithmsinsight.wordpress.com/graph-theory-2/a-star-in-general/implementing-a-star-to-solve-n-puzzle/ [Accessed 26 Aug. 2024]

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