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The Non-Existence of Algorithm Deciding Whether a Constructive Compact Topological Space is Simply Connected

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Abstract:
This paper discusses the non-existence of an algorithm Q always deciding whether a constructive compact topological space is simply connected or not, that is whether its fundamental group is trivial or not. We construct a program R that generates compact topological spaces, then we use R and Q to get an extension of an unextendible function to get the contradiction and finish the proof.

Keywords: constructive mathematics, topology, fundamental group, algorithm

1. Introduction

Topology, with its rich history and profound impact on mathematics and beyond, has evolved significantly since its inception in the late 19th century. While classical topology often relies on non-constructive proofs that establish the existence of certain structures without specifying how to construct them, the field of constructive topology has emerged to address this gap. This discipline emphasizes algorithmic and computational methods to explicitly build topological objects, offering a bridge between theoretical results and practical applications.

The primary work of constructive mathematics was done by Alan Turing [1,2]. After he gave the definition of computable numbers, constructive mathematics was developed by two different schools - Bishop and Bridges in the USA, and the others are Markov [3,4]and Shanin from Russia [5], Bishop and Bridges [6]. The difference between these two schools is that followers of Bishop and Bridges' Constructive Analysis do not allow themselves to use Markov's principle in their mathematics. And in this paper, we will discuss a problem in constructive topology.

2. Theory

2.1 Basic Topology

2.1.1 Compactness

We say a collection of subsets A is a covering of a space X, if the union of the elements of A is equal to X. It is said to be an open covering of X if these elements are all open subsets of X. A space X is called compact if every open covering of X has a finite sub-cover which means a finite subcollection also covers X. This definition is from Munkres [7].

2.1.2 ϵ-net

Given $\epsilon > 0$ and given a metric space (X,d), an ϵ-net $T_\epsilon \subset X$ is a finite set of points such that every $x \in X$ is within ϵ from a point of T_ϵ.

The following statement is well known.

Lemma

A closed subset of a complete and separable metric space (X,d) is compact iff for every $\epsilon > 0$, the finite ϵ-net of X does exists.

2.1.3 Constructive Topological Space

In constructive mathematics, a *constructive compact topological space* is defined as the closure of the union of algorithmically generated ϵ-nets for $\epsilon = \frac{1}{2^k}$, $k \in N$.

Remark: This gives a different object from the constructive version of a compact set given by open covers and the closed constructive interval [0,1] is not compact with respect to the open cover definition, but it is compact with respect to the ϵ-net definition [5], In the following discussion, we will use the ϵ-net definition.

2.1.4 Fundamental Group

Let X be a space and x_0 be a point of X. A path in X which begins at x_0 and ends at x_0 is called a *loop* that based at x_0. The set of path homotopy classes of loops based at x_0 equipped with the concatenation operation *, is called the fundamental group of X that relative to the base point x_0. We denoted it by $\pi_1(X, x_0)$.

2.1.5 Simply Connected Spaces

A space is called *simply* connected if its fundamental group is trivial.

2.2 Constructive Mathematics

2.2.1 Constructive Mathematics

Constructive mathematics is a branch of mathematics that emphasizes the construction of mathematical objects and proofs through explicit methods. In contrast to classical mathematics, which often relies on non-constructive techniques such as the law of excluded middle or the axiom of choice, constructive mathematics requires that mathematical objects be explicitly constructed and that proofs provide constructive methods for finding examples or solutions.

2.2.2 CRN

A Constructive Real Number (CRN) is a pair of algorithms (F,R). An algorithm F (the fundamental sequence) could transform natural numbers into rational numbers that are the members of a Cauchy sequence. Algorithm R (regulator of convergence) could transform positive rational numbers into natural numbers and it could guarantee the convergence in itself of the sequence F, so that for every positive rational $\epsilon > 0$ and every $m,n > R(\epsilon)$ we have $|F(m) - F(n)| < \epsilon$.

Definition: A CRN with a standard regulator is a CRN (F,R) that $R(n) = n, \forall n \in N$, every CRN is equivalent to a CRN with a standard regulator.

2.2.3 Computable Function

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A function $f:N\to N$, is called computable if there exists an algorithm or a mechanical procedure that, given any input $n\in N$, will compute $f(n)$ in a finite number of steps.
A constructive real function is an algorithm that transforming every CRN into a CRN taking equivalent CRNs to equivalent CRNs.
And the computable functions and constructive functions are truly the same from the space of all constructive real numbers CRNs to the space of all CRNs. And all the usual functions such as sine, logarithm etc. are constructive.

2.2.4 Extendible Program

Suppose there is a program H that could transfer positive integers to either 0 or 1. H is called extendible to all positive integers if there is another program E that reproduces the outputs of H whenever H halts, and program E produces some outputs for every positive integer.

2.2.5 Unextendible Algorithmic Functions

An extension of a function f is a function g, such that f is a restriction of g. And by a classical Theorem [8], we can find the computable function f defined on some of the positive integers that has no total extension.

3. Theorem

There is no algorithm that given an arbitrary constructive compact topological space can always tell if the space is simply connected or not.

4. Proof

Prove by contradiction. By Section 1.2.5 there exists a computable function P defined on some of positive integers that has no total extension. Assume that Q is an algorithm that can always tell us whether a constructive compact space X is simply connected or not, such that

$$Q(X)=\begin{cases}1, if X is simply connected\\0, if X is not simply connected\end{cases}$$

Then it is only necessary to find that there do exists an extension of the computable function P to deduce the contradiction.
R is an algorithm generating constructive compact constructive space. It together with $P(n)$ run at the same time. Now we introduce R in details. Denote X by the space [0,1]×[0,1]. For the first second, R generates an 1-net of X. Namely, the 1-net is denoted by $N(1)=\{(0,0),(0,1),(1,0),(1,1)\}$.

And then in the second, R generates an $\frac{1}{2}$-net denoted by $N(2)=\left\{(\frac{i}{2},\frac{j}{2}):i,j=0,1,2\right\}$.
And it keeps going on like this. When it is in the k-th second, R generates an $\frac{1}{2^{k-1}}$-net denoted by

$$N(k)=\left\{(\frac{m}{2^{k-1}},\frac{n}{2^{k-1}}):m,n=0,1,...,2^{k-1}\right\}.$$

There are two possible outputs when the program is running.
If $P(n)$ terminates at t-th second with some output, then R generates finitely many points in the square. Denote $X_n=\bigcup_{i=1}^{t}N(i)$, since X_n is finite, we can always remove a square with a side length of $\frac{1}{2^{t+1}}$ from X, which is denoted by

$$C(t)=\left\{(\frac{1}{2}-\frac{a}{2^{t+2}},\frac{1}{2}+\frac{b}{2^{t+2}}):a,b=1,3\right\}.$$

Then R will keep on generating ϵ-nets in $X\setminus C(t)$. That is, R outputs the space $X\setminus C(t)$.
If $P(n)$ never terminates, then R generates the epsilon nets for the whole square X and hence it will be the resulting compact set.
No matter what integer we input as a result of computing $P(n)$, R always outputs a space, which is a square with the center deleted and some extra points inside of it, so it is NOT simply connected.
In both cases we apply Q to $R(n)$, and get a new algorithm P. For integer n, if $P(n)$ terminates, then $R(n)$ is a square with a smaller square inside removed, whose fundamental group is nontrivial, hence P outputs 0; If $P(n)$ doesn't terminate, then $R(n)$ is just X, so P outputs $P(n)$. So P is an extension of P, contradicting "P is unextendible".

5. Corollary

There is no algorithm Q telling if the fundamental group of any arbitrary compact constructive topological space is a non-abelian free group with k generators.

6. Sketch of Proof

Suppose such an algorithm Q exists. Denote P by the unextendible in the proof above, we can construct compact metric topological spaces $R(n)$ in the same way as above but just removing k disjoint rectangles as $P(n)$ terminates. Then we could get an extension of P by Q and R, hence the contradiction.

7. Conclusion

The conclusion is that there does not exist an algorithm detecting the simply-connectedness of any compact constructive topological space. By generalizing this conclusion, we find that it is also impossible to detect whether the fundamental group of compact topological space has some certain structure or not. This result demonstrates some limitations in computational approaches to topology. We anticipate that our findings will encourage further exploration of this subject and motivate mathematicians to re-evaluate traditional inquiries within the framework of CRN. Looking ahead, we aim to broaden our research to encompass more intricate topological characteristics associated with CRN.

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