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Effect System and Reference Frames in Rubik's Cube and Megaminx

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Abstract:

The two basic puzzles in Rubik's cube can be summarized as recognizing solvable configurations and finding a common solution for valid configurations. Indeed, in the process of solving the first puzzle, we are required to come up with a common solution. The mathematical structure of moves in Rubik's cube is also famous. This paper clarifies the difference between process and essence through an original concept: effect, and redefines the operation group G in a strict way. It also corrects a widespread mistake in pinpointing the mapping type between the operation group G and permutation groups, which appears in some former papers. The effect system grasps the essence of operations, introduces reference frames in physics through pure mathematics, and specifically facilitates the solution for the basic puzzles in Rubik's cube and megaminx. The effect system has the potential to be utilized in physically symmetric structures with moves that permute mathematically analyzable configurations.

Keywords: Rubik's cube, Megaminx, Group theory, Valid configuration, Solution, Reference frame, Effect

1. Basic knowledge

In this part, we will cover the necessary mathematics for this paper. We will list the well-defined definitions and theroems, but not give proofs. If you are not familiar with some knowledge, access to the textbooks about abstract algebra, for example, *Abstract Algebra* from David S.Dummit and Richard M.Foote.

Definition1.1(Binary Operation) A binary operation * is a kind of operation which turns two elements into a new element(can be one of the original two), written as: a*b=c.

Definition1.2(Group) A group G is a set with a binary operation (*) which can work on any two elements in the set, satisfying four properties:

1. Closure: For any two elements a and b in the set, a*b is still in the set(a and b can be the same element)
2. Associativity: For any three elements a, b and c, (a*b)*c=a*(b*c).
3. Identity: There exists an element e in the set which satisfies e*a=a*e=a for any element a. (Usually we denote the identity in a group by e)
4. Invertibility : For any element a in the set, there exists an element b which satisfies a*b=b*a=e. (Usually such b is called an inverse of a)

Theorem1.1(Uniqueness of inverse) An element in a group only has one inverse.

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Definition1.3(Permutation) A permutation is a rearrangement of the order of elements in a set.

Definition1.4(Cycle notation of permutation) Any permutation can be denoted by a cycle. Specifically, if a permutation σ *turns n elements assigned in the order* (m_1,m_2,m_3,...,m_n) to an order (m_{i1},m_{i2},m_{i3},...,m_{in}), where i1 to in are a sequence of 1 to n, then σ is written as (m_1 $m_{i1...}$) (m_k m_{ik}...)... . Here we say that m_1 goes to m_{i1}, m_2 goes to m_{i2}, and so on; we start the cycle notation from m_1, and end the first cycle by the element which is turned into m_1, if this cycle hasn't included all the elements, we continue to the second cycle which begins with the leftmost element that is not included yet. This process ends when all the elements are included in the cycles.

Definition1.5(Product of permutations) Two permutations σ_1 and σ_2's product is still a permutation which satisfies $(\sigma_1\sigma_2)(i)=(\sigma_1)(\sigma_2(i))$ for any element i in the corresponding set.

Theorem1.2(Decomposition of permutations) Each permutation, written by definition1.4 as a product of (long) cycles, can be written in a product of some 2-cycles(cycles which include 2 elements), here " product of 2-cycles" refers to the product of permutations which are denoted by 2-cycles.

Theorem1.3(Permutation group) The set of permutations working on n(n is a positive integer) elements with a binary operation as the product defined in Definition1.5 forms a group. We call this group a permutation group, denoted by S_n.

Theorem1.4(Fixness of decomposition):In any notation of 2-cycles' product representing the same permutation, the amount of 2-cycles has the same parity.

Definition1.6(Signature) We designate the permutations which can be decomposed into an odd number of 2-cycles to have signature -1, the permutations which can be decomposed into even number of 2-cycles to have signature 1.

Definition1.7(Subgroup) If G is a group, H is a subset of G and the elements in H follow the same binary operation in G, additionally, if H satisfies the four properties of a group, then H is called a subgroup of G.

Definition1.8(Homomorphism)If G(The binary operation is #) and G'(The binary operation is *) are two groups, there exists a mapping φ which satisfies $\varphi(g_1)*\varphi(g_2) = \varphi(g_1\#g_2)$, where the mapping is from G to G', g_1 and g_2 are any two elements in group G, then φ is called a homomorphism(from G to G'). (When we are talking about a mapping from a group or to a group(or from and to a group), we consider the mapping from set to set, regardless of the binary operation in a group.)

Definition1.9(Kernel) We call a set the kernel of a mapping if this mapping is a homomorphism from a group G to a group G' and this set exactly contains all the elements that are mapped to the identity in group G'. The kernel of a mapping φ is sometimes written as kerφ.

Definition1.10(Restrict) If there is a mapping φ from a set(or a group) A to a set(or a group) B and there is a subset A' of set A, then we call the mapping φ' from A' to B which follows the same rules of φ a restrict on φ.

As an alternative notation, $\varphi|_{A'}$ refers to the same restrict on φ.

Definition1.11(A_k) A_k is the subgroup of S_k consisting of all the even permutations(permutations with signature 1) in S_k.

Definition1.12(Generator) A set of elements are called a generator of a group if and only if these elements and their inverses' all possible finite products(By the binary operation; By finite elements) cover all the elements in this group.

Theorem1.5(Generators of A_k and S_k) The set of all the 3-cycles in A_k is a generator of A_k, the set of all the 2-cycles in S_k is a generator of S_k.

Definition1.13(Group action) We say a group G acts on a set A if for any a in A and any g in G, there is an operation * that a * g is still an element in A and this operation satisfies the following two properties:

1. $(a*g_1)*g_2=a*(g_1g_2)$ for any g_1, any g_2 2. a*e=a

(Here g_1g_2 is the product in G by binary operation)

Definition1.14(Orbit) If G acts on A, then the orbit of any element a in A is: $\{a*g|g \in G\}$.

Theorem1.6 Suppose a finite group G acts on a set A, and let S be a generator of G. Let P be a property such that the following is true: Whenever $a \in A$ satisfies P and $s \in S$, a * s also satisfies P.

Then, if $a' \in A$ satisfies P, every element in the orbit of a' also satisfies P.

The proof for theorem1.6 can be found in [1]. Theorem1.6 will be useful when we analyze Rubik's cube and megaminx.

2. Mathematization of Rubik's Cube

2.1 Rubik's Cube

Rubik's cube is a symmetric structure with six big faces and 26 small cubes, which form a 3×3×3 cube (The kernel space is empty). (Fig.2.1) There are 6 center cubes, each has only one face with color; 12 edge cubes, each has two faces with color; 8 corner cubes, each has 3 faces with color. The upper big face is denoted by U, the right big face is denoted by R, the left big face is denoted by L, the frontal big face is denoted by F, the bottom big face is denoted by D and the back big face is denoted by B.

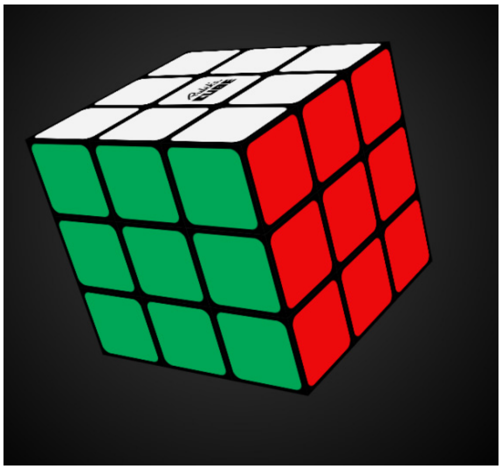

Figure2.1 A Rubik's cube view from red face

If a Rubik's cube looks the same as Fig.2.1, we say that it is solved. Each big face can be rotated, but other rotations are not allowed, such as rotating a slice between two big faces. The six big faces correspond to six colors:white, blue, green, red, orange and yellow because the center cubes are always fixed in position. When consider rotations, a capital refers to the rotation of the corresponding big face clockwise by 90 degrees, a capital with power -1 refers to the rotation of the corresponding big face counterclockwise by 90 degrees(Here "clockwise" is local, we mean the clockwise rotation seen from respective big face). As a custom, we call a sequence of rotations which is viable or e(the element of operating nothing) a move. Now we call U, U^{-1},L,L^{-1},R,R^{-1},F,F^{-1}, B,B^{-1},D,D^{-1} the 12 single moves. It is a commen sense that any move can be denoted by a sequence of single moves, except e. One thing to clarify is that moves are defined by process------the sequence of single moves or the process of doing nothing, for example, RR^{-1} does not equal to $R^{-1}R$. The definition of moves is a pure reflection of physical world, which is easy to understand. However, the set of moves can not form a group with a binary operation "stacking in time" because every element except e lacks its inverse. Although in some lecture notes, Janet Chen solved this vaguely by arguing that all the moves which result the Rubik's cube in the same configuration are defined the same [1], then what is move, and what if the initial configurations are not the same? (You can simply regard configuration as the setting of small cubes' position and orientation, the strict definition is in chapter 2.2) Move is move, it is a sequence of rotations, a reflection of process. Nonetheless, the lecture notes' idea is correct, it aims to mix all the moves which make the same change, or the same effect. One of this paper's focuses is to address this issue clearly and nicely, you will see in chapter 2.4.

Before discussing configurations, we need to introduce some basic notation which tracks some features of a Rubik's cube. As a tradition [2], we use lowercase letters to denote the position of corner cubes and edge cubes.

For example, dbr refers to the corner cube which has a face in D, a face in B and a face in R; uf refers to the edge cubes which has a face in U and a face in F. However, when consider corner cubes' position and edge cubes' position seperately, this paper will sometimes adapt an easier notation(numbers):

Corner cubes:

1:ulf 2:urf 3:urb 4:ulb 5:dlb 6:dlf 7:drf 8:drb

Edge cubes:

1:ub 2:ur 3:uf 4:ul 5:bl 6:br 7:fr 8:fl 9:db 10:dr 11:df 12:dl

In order to denote the orientation of corner cubes and edge cubes in different positions, we think of a consistent trace "mark" and an unchangeable evaluation system.

Imagine using a mark pen to write numbers 0,1 and 2 on each corner cube when the Rubik's cube is original (solved). We write 0 on each upper face of the four upper cubes(1,2,3,4) and each lower face of the four bottom cubes, then we designate 1 and 2 respectively to each corner cube. When designating, we assign 1 and 2 locally clockwise(When you look from the outer world to a specific corner cube, you assign 1 and 2 clockwise after 0. It does not matter if you look from any of the perspective where you can see the corner cube, because"clockwise"in this situation is the same). That is, following a locally clockwise order, we mark 0,1,2 on each cube. One thing important is that these marks move with the corner cubes. Now, consider a specific assignment of small cubes' position and orientation which we can achieve through moves, we use x_1 to represent the mark that appears on ulf's u face after moving to this specific assignment from the original, we use x_2 to represent the mark that appears on urf's u face after moving to this specific assignment from the original, and so on(Here x_1 to x_8 correspond to 1 to 8 respectively).

Similarly, we can use marks accompanying with cubes and an unchangeable evaluation system which tracks specific places' values to show edge cubes' orientation. Imagine that there exists an orginal Rubik's cube with all small cubes in the right position and have the right orientation. We mark 0 on the u face of ub, the u face of ur, the u face of uf, the u face of ul, the b face of bl, the b face of br, the f face of fr, the f face of fl, the d face of db, the d face of dr, the d face of df and the d face of dl. Other 12 faces on edge cubes are marked 1. Now we can consider any specific assignment of small cubes' position and orientation (that can be moved from the original), y_1 to y_{12} track the values on the places which originally have 0s.

To extend the definition to the assignment of cubes which

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we can not move to from original assignment, we imagine that after marking all the values, we split all the corner cubes along with edge cubes and reassemble them to an assignment which can not be achieved through moves, then we still use x_1 to x_8 and y_1 to y_{12} to track the values.

2.2 Configuration

Configurations in Rubik's cube refers to the settings of small cubes. After all, we can only change the position and orientation of small cubes. In order to facilitate our discussion in following chapters, we now consider all the possible configurations of a Rubik's cube after splitting off all the corne cubes along with edge cubes and reassembling them on. The configuration of a Rubik's cube is denoted by(σ,τ,x,y), corresponding to four dimensions: the position of corner cubes, represented by σ; the position of edge cubes, represented by τ; the orientation of corner cubes, represented by x; the orientation of edge cubes, represented by y. It is clear that a configuration is exactly determined by these four dimensions. On the one hand, if these four dimensions are known, we have exactly one way to assemble the small cubes. On the other hand, once we know the configuration of a Rubik's cube, we know the information of these four dimensions. By inheritng the notation from Janet Chen[1], we define the four dimensions' mathematical expression well in the following way:

Suppose any configuration $a = (\sigma,\tau,x,y)$.

1. σ(corner cubes' position):

σ: This is a permutation, we define it as the permutation which turns the 8 corner cubes from original position to the position in a(The original position is:1 2 3 4 5 6 7 8; the position in a is the number sequence of corner cubes in a, from ulf to dbr, here number means the original number for a cube).This permutation exactly describes the position of corner cubes in configuration a.

2. τ(edge cubes' position):

τ: This is a permutation, we define it as the permutation which turns the 12 corner cubes from original position to the position in a(The original position is:1 2 3 4 5 6 7 8 9 10 11 12; the position in a is the number sequence of edge cubes in a, from ub to dl, here number means the original number of a cube). This permutation eaxctly describes the position of edge cubes in configuration a.

3. x(corner cubes' orientation):

x: $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ Here x_1 to x_8 refers to the 8 values in configuration a, as we defined in chapter2.1.

4. y(edge cubes' orientation):

y: $y=(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9,y_{10},y_{11},y_{12})$ Here y_1 to y_{12} refers to the 12 values in configuration a, as we defined in chapter2.1.

2.3 Effect

Effect is commonly used in daily life, but in this paper, we will define effect through pure mathematics. It will help us clarify the group G(operation group) which acts on the set of configurations, and promote the construction of theorems based on reference frames.

By some observation, we can notice that many moves essentially do the same thing. For example, R and R^5(We usually use power to denote repetitive single moves in a row)are essentially the same because they both exactly rotate the right big face clockwise by 90 degrees. We define effect as the abstract expression of such essence.

Here is the definition:

Definition2.1(Effect) If a move M turns configuration $a = (\sigma,\tau,x,y)$ to configuration $a' = (\sigma',\tau',x',y')$, then this move's effect is $ef(M)=(\sigma'\sigma^{-1},\tau'\tau^{-1},X,Y)$.

Here $x'=(x'_1,x'_2,x'_3,x'_4,x'_5,x'_6,x'_7,x'_8)$; $y'=(y'_1,y'_2,y'_3,y'_4,y'_5,y'_6,y'_7,y'_8,y'_9,y'_{10},y'_{11},y'_{12})$; $X=((x'_{\sigma'\sigma^{-1}(1)}-x_1)\bmod 3,(x'_{\sigma'\sigma^{-1}(2)}-x_2)\bmod 3,\ldots,(x'_{\sigma'\sigma^{-1}(8)}-x_8)\bmod 3)$; $Y=((y'_{\tau'\tau^{-1}(1)}-y_1)\bmod 2,(y'_{\tau'\tau^{-1}(2)}-y_2)\bmod 2,\ldots,(y'_{\tau'\tau^{-1}(12)}-y_{12})\bmod 2)$.

In the following discussion, we will use $ef(M)_1$ to denote corresponding $\sigma'\sigma^{-1}$, $ef(M)_2$ to denote corresponding $\tau'\tau^{-1}$, $ef(M)_3$ to denote corresponding X, $ef(M)_4$ to denote corresponding Y. We can deduce a theorem which connects effect with familiar expression.

Theorem2.1 For any move M which turns configuration a to a', $ef(M)_1$ is the permutation of M on the position of corner cubes;$ef(M)_2$ is the permutation of M on the position of edge cubes; assume $ef(M)_3$'s kth value is m, 120m is the degree which we need to rotate a corner cube counterclockwise after it is moved from the position in configuration a to the position in configuration a' while keeping the tracked face in configuration a still a tracked face; assume $ef(M)_4$'s lth value is n, 180n is the degree which we need to rotate a edge cube counterclockwise after it is moved from the position in configuration a to the position in configuration a' while keeping the tracked face in configuration a still a tracked face.

(Here tracked face means the face which we record its value in x or y; k can be any integer among 1 to 8; l can be any integer among 1 to 12)

Proof. Let the permutation of position of corner cubes be σ_{corner}. Because $\sigma_{corner}\sigma(i) = \sigma'(i)$ when we take i from 1 to 8, $\sigma_{corner}\sigma = \sigma'$. Therefore, $\sigma_{corner} = \sigma'\sigma^{-1}$. Similarly, we get $\tau_{corner} = \tau'\tau^{-1}$, thus we finish half the proof. Now consider corner cubes' orientation. Assume that the original tracked face on the corner cube i is turned to a specific position in configuration a'(i can be any integer from 1 to 8). Now x_i is just the value on this initially tracked face. If this face in a' is 120k degrees behind the tracked face on the same cube(k may be 0 or 1 or 2; “behind” means we need to rotate the corner cube locally clockwise by some degrees to turn a face into the tracked face), then $x'_{\sigma'\sigma\text{-}1(i)} = (x_i+k)$ mod3. Thereby, $(x'_{\sigma'\sigma\text{-}1(i)}-x_i)$mod3 should be k. Because rotating a face from a specific place to the tracked place on the cube is just the inverse process of rotating the same cubes’ tracked face to the specific place, we finish our proof about $ef(M)_3$. Similarly, we obtain that the $ef(M)_4$ part in the theorem is right.

Through theorem2.1, we know that a effect should exactly describes what a move do to the Rubik’s cube,

regardless of the initial configuration.

Theorem2.2 If A,B are two moves and they do the same thing, then ef(A)=ef(B).

(Do the same thing means that A and B send any pair of two cubes which locate in the same position of the two initial configurations to the same position of the two final configurations and rotate them in the same way.)

Proof. This is a direct corollary of theorem2.1.

By theorem2.2, we know that we can refer to any effect by merely writing it as ef(M), here M is a move which has the effect we want. That is, we don’t need to consider the initial configurations because the same move acting on different initial configurations should do the same thing. In the following text, we may stilll mention the initial configurations and the final configurations of a move, mainly for denoting some effects.

For the sake of systematicness, we define the product of effects.

Definition2.2(Multiplication of effect) If a move A turns configuration $a=(\sigma_a,\tau_a,x_a,y_a)$ to configuration $a'=(\sigma_a',\tau_a',x_a',y_a')$, a move B turns configuration $b=(\sigma_b,\tau_b,x_b,y_b)$ to configuration $b'=(\sigma_b',\tau_b',x_b',y_b')$, then $ef(A)ef(B)=(\sigma_b'\sigma_b^{-1}\sigma_a'\sigma_a^{-1},\tau_b'\tau_b^{-1}\tau_a'\tau_a^{-1},X_{ab},Y_{ab})$, where $X_{ab}=((X_{b(\sigma a'\sigma a\text{-}1)(1)}+X_{a1})\text{mod}3,(X_{b(\sigma a'\sigma a\text{-}1)(2)}+X_{a2})\text{mod}3,\ldots,(X_{b(\sigma a'\sigma a\text{-}1)(8)}+X_{a8})\text{mod}3)$, $Y_{ab}=((Y_{b(\tau a'\tau a\text{-}1)(1)}+Y_{a1})\text{mod}2,(Y_{b(\tau a'\tau a\text{-}1)(2)}+Y_{a2})\text{mod}2,\ldots,(Y_{b(\tau a'\tau a\text{-}1)(12)}+Y_{a12})\text{mod}2)$, X_{ai} represents the ith value in X_a, X_{bj} refers to the jth value in X_b, Y_{ak} refers to the kth value in Y_a, Y_{bl} refers to the lth value in Y_b. (We denote $ef(A)_3$ by X_a, $ef(A)_4$ by Y_a, $ef(B)_3$ by X_b, $ef(B)_4$ by Y_b)

Usually we only consider the overall effect of a move A followed by a move B. Thus, to verify that Definition2.2 defines the product of effect well, we only need to show that ef(AB)=ef(A)ef(B).

Theorem2.3 ef(AB)=ef(A)ef(B)

Proof. $\sigma_b'\sigma_b^{-1}\sigma_a'\sigma_a^{-1}$ is obviously the permutation of position of corner cubes by AB considering any corner cube i, $\tau_b'\tau_b^{-1}\tau_a'\tau_a^{-1}$ is also obviously the permutation of position of edge cubes by AB. For any interger i among 1 to 8, because $(X_{b(\sigma a'\sigma a\text{-}1)(i)}+X_{ai})\text{mod}3 = ((x_b'{}_{(\sigma b'\sigma b\text{-}1\sigma a'\sigma a\text{-}1)(i)}-x_{b(\sigma a'\sigma a\text{-}1)(i)})\text{mod}3+(x_a'{}_{(\sigma a'\sigma a\text{-}1)(i)}-x_{ai})\text{mod}3)\text{mod}3 = (x_b'{}_{(\sigma b'\sigma b\text{-}1\sigma a'\sigma a\text{-}1)(i)}-x_{b(\sigma a'\sigma a\text{-}1)(i)}+x_a'{}_{(\sigma a'\sigma a\text{-}1)(i)}-x_{ai})\text{mod}3 = ef(AB)_{3i}$ when $x_{b(\sigma a'\sigma a\text{-}1)(i)} = x_a'{}_{(\sigma a'\sigma a\text{-}1)(i)}$, we only need to consider the situation of $x_{b(\sigma a'\sigma a\text{-}1)(i)}\neq x_a'{}_{(\sigma a'\sigma a\text{-}1)(i)}$. ($x_{ai}$ refers to the ith value in x_a, similar for $x_a'{}_i$; $ef(AB)_{3i}$ refers to the ith value in $ef(AB)_3$)

By theorem2.2, we know that the equation still holds true when $x_{b(\sigma a'\sigma a\text{-}1)(i)}\neq x_a'{}_{(\sigma a'\sigma a\text{-}1)(i)}$ because$(x_b'{}_{(\sigma b'\sigma b\text{-}1\sigma a'\sigma a\text{-}1)(i)}- x_{b(\sigma a'\sigma a\text{-}1)(i)})$mod3 and $(x_a'{}_{(\sigma a'\sigma a\text{-}1)(i)}-x_{ai})$mod3 remain the same. Therefore, $ef(AB)_3=X_{ab}$. Similarly, $ef(AB)_4=Y_{ab}$, which finishes our proof.

Theorem2.3 also shows that the product of effects is still an effect.

Theorem2.4 The multiplication of effect satisfies associativity.

Proof. Suppose $ef(M_1)$, $ef(M_2)$, $ef(M_3)$ are any three effects. Then it suffices to show that $(ef(M_1)ef(M_2))ef(M_3)= ef(M_1)(ef(M_2)ef(M_3))$. Firstly, $((ef(M_1)ef(M_2))ef(M_3))_1=(ef(M_1)(ef(M_2)ef(M_3)))_1$ and $((ef(M_1)ef(M_2))ef(M_3))_2= (ef(M_1)(ef(M_2)ef(M_3)))_2$ because the multiplication of permutations satisfies associativity. Secondly, $((ef(M_1)\ ef(M_2))ef(M_3))_3=(ef(M_1)(ef(M_2)ef(M_3)))_3$. Assume that M_1 turns a configuration $(\sigma_{m1},\tau_{m1},x_{m1},y_{m1})$ to $(\sigma_{m1}',\tau_{m1}',x_{m1}',y_{m1}')$,$M_2$ turns $(\sigma_{m2},\tau_{m2},x_{m2},y_{m2})$ to $(\sigma_{m2}',\tau_{m2}',x_{m2}',y_{m2}')$,$M_3$ turns $(\sigma_{m3},\tau_{m3},x_{m3},y_{m3})$ to $(\sigma_{m3}',\tau_{m3}',x_{m3}',y_{m3}')$,$M_2M_3$ turns $(\sigma_{m2},\tau_{m2},x_{m2},y_{m2})$ to $(\sigma_{m2}^*,\tau_{m2}^*,x_{m2}^*,y_{m2}^*)$.Then it suffices ti show that:$X_{m2}^*{}_{(\sigma_{m2}^*,\tau_{m2}^*,x_{m2}^*,y_{m2}^*)(i)}-X_{m2(\sigma_{m2},\tau_{m2},x_{m2},y_{m2})(i)}\,\Xi\,(X_{m3}'{}_{(\sigma_{m3}'\sigma_{m3}^{-1}\sigma_{m2}'\sigma_{m2}^{-1}\sigma_{m1}'\sigma_{m1}^{-1})(i)}-X_{m3(\sigma_{m2}'\sigma_{m2}^{-1}\sigma_{m1}'\sigma_{m1}^{-1})(i)})+(X_{m2}'{}_{(\sigma_{m2}'\sigma_{m2}^{-1}\sigma_{m1}'\sigma_{m1}^{-1})(i)}-X_{m2(\sigma_{m1}'\sigma_{m1}^{-1})(i)})$mod(3) because the three differences (LHS, RHS’s two brackets)won’t change with different asssumptions of initial configirations。 However, the equation should be true if we replace$(\sigma_{m1}'\sigma_{m1}^{-1})$(i)by j. Similarly, we can prove that $((ef(M_1)\ ef(M_2))ef(M_3))_4=(ef(M_1)(ef(M_2)ef(M_3)))_4$.

Theorem2.5 $ef(M_1M_2M_3\ldots M_n)=ef(M_1)ef(M_2)ef(M_3)\ldots ef(M_n)$

Proof. $ef(M_1M_2M_3\ldots M_n)=ef(M_1M_2\ldots M_{n-1})ef(M_n)=ef(M_1M_2\ldots M_{n-2})ef(M_{n-1})ef(M_n)=\ldots=ef(M_1)ef(M_2)ef(M_3)\ldots ef(M_n)$.

2.4 Operation Group

We can notice that a move has only one effect, but an effect is always owned by countless moves. For example, ef(R)=((2 3 8 7),(2 6 10 7),(0,2,1,0,0,0,1,2),(0,0,0,0,0,0,0,0,0,0,0,0))=ef(R^5)=ef(R^9)=... We now assign a name for a set of moves which have the same effect.

Definition2.3(Operation) An operation is a set of moves which have the same effect.

If an operation has a move M in it, we denote this operation by [M].

Now we can address the issue in chapter2.1 that moves

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can not form a group under strict definition because operations can. First we define the binary operation for operations.

Definition2.4(Product of operation) If M_1 and M_2 are two moves, then$[M_1][M_2]=[M_1M_2]$.

We designate the binary operation for operations as the product in Definition2.4.

Theorem2.6 All the operations in a Rubik's cube form a group. (We take the product of operations as the binary operation)

Proof.

Closure: By definition, we know the product of two operations is still an operation.

Associativity: $[(M_1M_2)M_3]=[M_1(M_2M_3)]$ because $ef((M_1M_2)M_3)=ef(M_1(M_2M_3))$. Therefore,

$([M_1][M_2])[M_3]=[M_1M_2][M_3]=[(M_1M_2)M_3]=[M_1(M_2M_3)]=[M_1][M_2M_3]=[M_1]([M_2]\ [M_3])$.

Identity: The identity can be [e]. For any operation [M], [e][M]=[M]=[M][e].

Invertibility: For any operation [M], where $M=m_1m_2...m_n$(m_1 to m_n are single moves), we define $M^{-1}=m_n^{-1}m_{n-1}^{-1}...\ m_1^{-1}$, (For any m_i from m_1 to m_n, m_i^{-1} is the single move which rotate the same face as m_i in opposite direction) then

$[M^{-1}][M]=[M^{-1}][M]=[e]$.

Indeed, the operation group is a finite group because the amount of possible effects is limited. We will call this finite group as G in following parts. In fact, former researchers have found this group but lack of strict definition, because they have not described the effects of moves so clearly[3]. When they define the operation group G, they could only vaguely argue that "we call the moves which result the Rubik's cube in same configuration the same move". If the initial configurations are not the same, then how can this statement work? Granted that we can say "we call the moves which do the same thing the same move", how can we define "do the same thing"? We can only explain " do the same thing" as changing the position and orientation of cubes in the same way, which is exactly the concept that "two effects are the same". Thus, the concept of effect is a must in clarifying the group G for Rubik's cube.

Effect is not only useful in clarifying group G. Effect itself expresses what a move does essentially and can be applied in clearly describing some mappings(Theorem2.7) and strictly deducing some algorithms(chapter3.1).

2.5 Homomorphisms from G

There are some homomorphisms from operation group G to other groups. However, a widespread mistake is that the mapping from G to S_8 is a homomorphism when we map each operation(In these papers operation is called "move") to the corresponding permutation of position of corner cubes[4]. The reason is that the binary operation defined in S_8 doesn't align with "stacking in time order". We need to create a new group named S_8' which has the same element as S_8 but a binary operation in a inverse order to form the homomorphism we want.

Definition2.5 S_k' is the group with the same elements of S_k but a binary operation * defined as: $\sigma_a * \sigma_b=\sigma_b\sigma_a$.

(k may be any positive integer, σ_a and σ_b can be any two permutations in S_k')

Theorem2.7 The mapping from G to S_8': φ_{corner} (for a move M, $\varphi_{corner}([M])=ef(M)_1$) is a homomorphism.

Proof. $\varphi_{corner}([M_1])*\varphi_{corner}([M_2])=ef(M_2)_1ef(M_1)_1=ef(M_1M_2)_1=\varphi_{corner}([M_1M_2])$.

(By theorem2.1, we know that $(ef(M_2)_1ef(M_1)_1)(i)=(ef(M_1M_2)_1)(i)$ for any i among 1 to 8, thus $ef(M_2)_1ef(M_1)_1=ef(M_1M_2)_1$)

3. Reference Frames and Valid configurations

In physics, we know that moves are relative because we can adapt different reference frames. This idea can be applied in the Rubik's cube. This chapter will introduce how this idea works and offer strict proofs. After that, some application of this in analyzing valid configurations will be discussed.

3.1 Reference Frames

As a commen sense, if we can make some specific changes to the Rubik's cube, such as flip two adjacent corner cubes at the same time without disturbing the position of all the cubes, then we can do the same thing on another two adjacent corner cubes by looking the whole Rubik's cube in a new way. That is, when we look from another perspective(Indeed change a reference frame), then we can do relatively the same thing as before because "similar" viewed from different reference frames can be "same". However, how can we prove "do the same thing" mathematically? The answer is to grasp the essence of moves ------ effects.

Definition3.1(24 Reference frames) In a Rubik's cube, we define the basic reference frames to be the 24 viewing it from 6 faces, with the corresponding 4 directions looking each face. (In following discussion, we will simply use "reference frame" to replace "basic reference frame")

In a new reference frame, we should define configuration and effect in a new way.

Theorem3.1 If there exists a move with a specific effect in a reference frame, then there exists a move with that effect in any reference frame.

Proof. We can prove this by decomposition. Assume the known move $M=m_1m_2m_3...m_n$, where m_1 to m_n are single moves. By theorem2.5, $ef(M)=ef(m_1)ef(m_2)ef(m_3)...ef(m_n)$. Now construct a move M' in the another reference frame. We require that M' is written as $m_1m_2...m_n$ in this reference frame. Thus, ef(M') in this reference frame is the same as ef(M) in the first reference frame.

Theorem3.1 implies that if there exists a move which does some thing in a reference frame, there should exist a move which does the same thing within any another reference frame because effects exactly correspond to what moves do(Theorem2.1).

Theorem3.2(Extension of moves) If there exists a move which does some change to the Rubik's cube within a reference frame, then there exists moves in the same reference frame for similar changes. (Here two similar changes means that when we view the second one from another reference frame, the change becomes the same as the first one)

Proof. The single moves in any reference frame are still single moves in other reference frames.

Theorem3.2 obviously has a value of application. An example is given in Fig3.2.

3.2 Valid Configuration of Rubik's Cube

As we mentioned in the abstract, the condition that configurations are valid has a sufficient and necessary condition. (Here valid means solvable)

Theorem3.3 A configuration a = (σ,τ,x,y) is valid if and only if sgnσ = sgnτ, $\Sigma x_i \equiv 0 (\mathrm{mod}3)$, and $\Sigma y_i \equiv 0$ (mod2).

This theorem can be proved by theorem1.6 with the help of generators. This paper will omit some discussions and apply theorem3.3 to prove the key facts needed for proving theorem3.3[1]. We will illustrate the process of extending a single move with a specific effect to some similar moves, which shows how the facts are proved in our way.

Fact3.1 Any three corner cubes' position can be cycled by move without changing the position of other corner cubes.

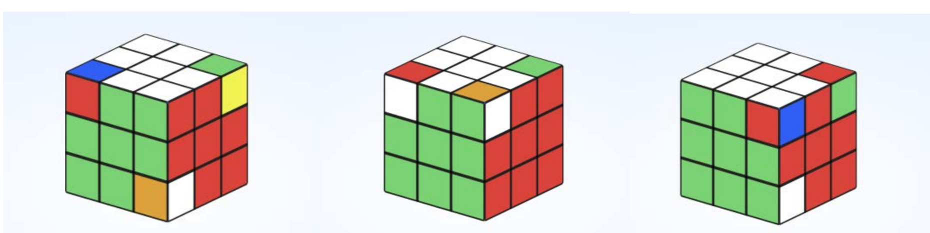

Figure3.1

*The first configuration can be restored by $F^{-1}UBU^{-1}FUB^{-1}U^{-1}$, the second configuration is the same situation within another reference frame. The third configuration is an instance that we can restore by combining the process restoring the first cofiguration and the process restoring the second configuration.

Fact3. 2 Any two corner cubes can be flipped by move without changing the position of all corner cubes and the orientation of other corner cubes.

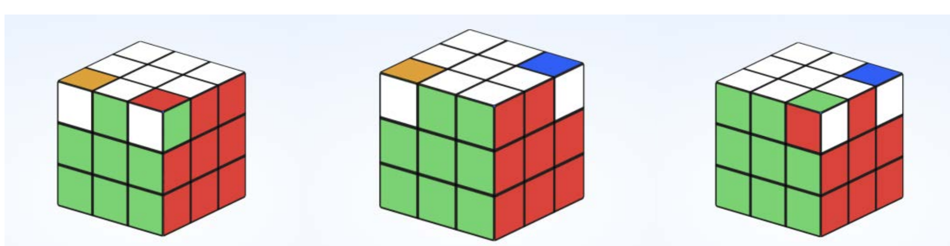

Figure3.2

*The first configuration can be restored by $(LD^{-1}L^{-1}F^{-1}D^{-1}FUF^{-1}DFLDL^{-1}U^{-1})^2$, the second configuration is the same situation within another reference frame. The third configuration is an instance that we can restore by combining the process restoring the first configuration and the process restoring the second configuration.

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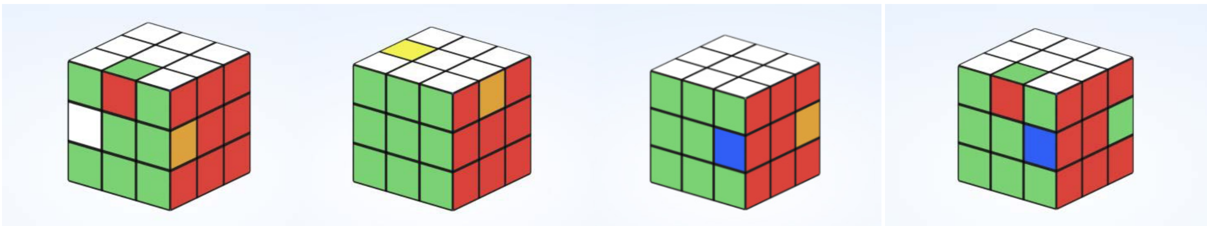

Fact3.3 Any three edge cubes' position can be cycled by move without changing corner cubes and the position of other edge cubes.

Figure3.3

*The first configuration can be restored by $ULLU^{-1}F^{-1}D^{-1}B^{-1}LLBDF$, the second configuration can be restored by $LR^{-1}FFL^{-1}RUU$. The fourth configuration is the combinition of the restoration method in the first configuration and the one in the third configuration, which is just the situation of the second configuration within a different reference frame.

Fact3.4 Any two edge cubes can be flipped by move without chaning their position and other cubes

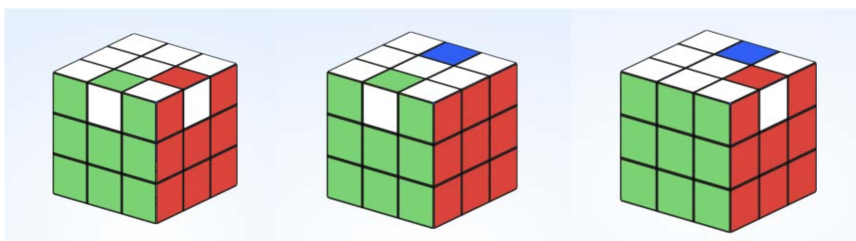

Figure3.4

*The first configuration can be restored by $FRBLUL^{-1}UB^{-1}R^{-1}F^{-1}L^{-1}U^{-1}LU^{-1}$. The second configuration is the same situation within a different reference frame. By combinition, we know that the third one can be restored.

3.3 Valid Configuration of Megaminx

To analyze megaminx, we will inherit effect system and reference frames from Rubik's cube. When defining configurations, we assign 0,1,2 to corner cubes and 0,1 to edge cubes locally clockwise. Based on the definition of condiguration, effect system is constructed with 60 basic reference frames. A conclusion is that theorem3.2 holds true if we replace Rubik's cube by megaminx.

Similar to chapter3.2, we will highlight the four facts which are the key for proving theorem3.4[5]. We will show how we extend certain moves through reference frames. The mechanism is exactly the same as how we extend moves in Rubik's cube. This paper will list the specific moves which restore the first figures.

Theorem3.4 A configuration a = (σ,τ,x,y) is valid if and only if $\text{sgn}\sigma = \text{sgn}\tau = 1$, $\Sigma x_i \equiv 0 (\text{mod} 3)$, and $\Sigma y_i \equiv 0$ (mod2).

Fact3.5 Any three corner pieces' position can be cycled by move without changing the position of other corner pieces.

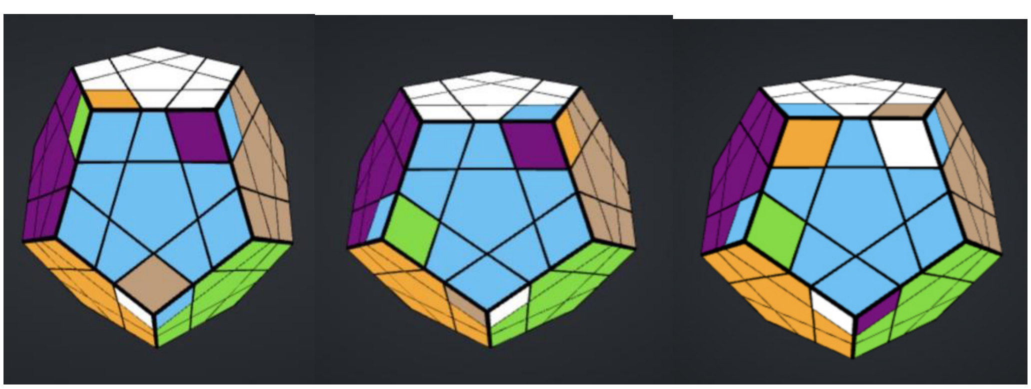

Figure3.5

*The move which restores the first figure is $D^{-1}FDA^{-1}D^{-1}F^{-1}DA$.(The letters correspond to the 72 degree clockwise turns of different faces. A letter with a power -1 means turning the face 72 degrees counterclockwise. Here A corresponds to the face with center cube white, B corresponds to the fae with center cube purple, C to blue, D to brown, E to orange, F to green. In other reference frames, the same letter refers to the relatively same position's face.)

Fact3.6 Any two corner pieces can be flipped by move without changing the position of all corner pieces and the orientation of other corner pieces.

Figure3.6

*The move which restores the first figure is $D^{-1}F^{-1}F^{-1}E^{-1}FDA^{-1}D^{-1}F^{-1}EFFDA$.

Fact3.7 Any three edge pieces' position can be cycled by move without changing corner pieces and the position of other edge pieces.

Figure3.7

*The move which restores the first figure is $D^{-1}D^{-1}F^{-1}B^{-1}DC^{-1}C^{-1}D^{-1}BA^{-1}B^{-1}DCCD^{-1}BAFDD$.

Fact3.8 Any two edge pieces can be flipped by move without chaning their position and other pieces.

Figure3.8

*The move which restores the first figure is $B^{-1}DC^{-1}C^{-1}EFC^{-1}C^{-1}D^{-1}BA^{-1}B^{-1}DCCF^{-1}E^{-1}CCD^{-1}BA$.

4. CONCLUSION

This paper introduces effect to denote the essence of moves. With the help of effect, it clarifies the structure of the operation group G in Rubik's cube with well defined elements and well defined binary operation. Besides, this paper corrects the mapping from G to S_8 and G to S_{12}, further deduces the homomorphism from G to S_8' and the homomorphism from G to S_{12}'. Finally, this paper strictly proves that all similar effects can be achieved if one effect is achieved within one reference frame through effect system, and apply this property in proving the key facts for analyzing valid configurations.

REFERENCE

[1] J. Chen, Group theory and the Rubik's cube, Lecture Notes.

ISSN 2959-6157

[2] T. Davis, Group theory via Rubik's cube, Lecture Notes.
[3] "result in the same configuration the same move...", J.Chen, Group theory and the Rubik's cube, Lecture Notes.
[4] "Similarly, we can define a homomorphism...), J.Chen, Group theory and the Rubik's cube, Lecture Notes.
[5] Xu G J, Gu D X. Solving Megaminx puzzle With Group Theory[J].