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Representation Theory of Symmetric Groups and Combinatorics

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Abstract:

This paper contains an introduction of representation theory of symmetric groups and several applications in Combinatorics. By using Hook length Formula, we then consider two specific shaped Standard Young Tableaux (SYT) where (*n, n*) shaped SYT has a connection with Catalan Numbers and the number of ways to fill a triangular formed SYT represents the largest irreducible representation of such symmetric group.

Keywords: Symmetric Groups; Standard Young Tableaux, SYT; Hook Length Formula; Irreducible Representations; Catalan Numbers

1. Introduction

In this paper, we discuss representation theory of symmetric groups. Starting from the fundamental concepts such as the bijections involved and the character table, where the latter indicates an intriguing fact that two different groups with the same cardinality, for instance dihedral group D_4 and quaternion group Q_8, can share the same character table without being isomorphic to each other.

Using the Standard Young Tableaux to represent a product group of symmetric groups, we are interested in calculating the number of fillings in a SYT, of a certain shape, which has already been proved in [Boo31] is exactly the dimension of the new irreducible representation of such a group. Theorem 3.4 is useful to derive the number of fillings of any kinds of SYT, where (*n, n*) SYT is exactly the Catalan Number *Cn*, and triangular one has the maximum dimension of irreducible representation of the corresponding symmetric group. However, formulating an expression for the latter case is looking complicated. Thus, an asymptotic approximation is derived by the inspiration of Stirling's Formula.

1.1 Preliminaries and Notations

In this subsection, we review some basic definitions in Representation Theory, and we can see [FH13] for more detail.

Definition 1.1. A *representation* of a group *G* (over a field $\mathbb{K}$) is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is a finite dimensional vector space (over $\mathbb{K}$).

Example 1.2. Let $V = \mathbb{C}^k$, then $\mathrm{GL(V)} = GL\left(\mathbb{C}^k\right) = GL_k\left(\mathbb{C}\right)$.

Definition 1.3. Let *V* be a representation of *G*. A *G-invariant subspace W* of *V* is a vector subspace of *V* such that $g \cdot w \in W$ for all $g \in G$ and $w \in W$.

Definition 1.4. A representation *V* of *G* is *irreducible* if and only if the only *G*-invariant subspaces of *V* are {0} and *V* itself.

1.2 Structure of the paper

In Section 2, we focus on representation of symmetric group *Sn*, where we study two bijections in Section 2.1, and then we give some examples of the character table of different groups in Section 2.2.

In Section 3, we move on to the applications in Combinatorics, starting from the Standard Young Tableaux (SYT), where the Hook length Formula (Theorem 3.7) introduced in [FRT54] tells us the number of fillings of the SYT with a specific shape λ. We then use it to derive the number of fillings of two specific shaped SYT in Section 3.2 and 3.3.

2. Representation Theory of Symmetric Group

We first recall that a *permutation* of $\{1,2,...,n\}$ is a bijection function

$f : \{1,2,...,n\} \to \{1,2,...,n\}$

and we write S_n for the set of all permutations.

2.1 Bijections

In this section, we want to show several beautiful bijections involved, we first review some knowledge about conjugacy classes and one can see [Gri07] to find more detail.

Definition 2.1. Consider a group G. Two elements a, $b \in G$ are conjugate to each other if there exist $g \in G$ such that $b = gag^{-1}$.

This relation defines an equivalence relation on G, and the resulting equivalence classes are called *conjugacy classes*.

Remark 2.2. If S is a set equipped with an equivalence relation ~, then every element of S is in one and only one equivalence class.

Corollary 2.3. The following three sets are in bijection with each other

1. conjugacy class in S_n
2. partitions of n
3. irreducible representations of S_n

Proof of $1 \leftrightarrow 2$. From Remark 2.2, equivalence classes give a partition on the set S. Therefore, our statement follows naturally by the note above. □

Remark 2.4. For the symmetric group Sn, a natural bijection exists between conjugacy classes and irreducible representations. However, $2 \leftrightarrow 3$ does not hold for a general group G.

2.2 Character table

Definition 2.5. A *character* of a representation ρV of G on a finite-dimensional complex vector space V is the map $\chi_V : G \to \mathbb{C}$ defined by $\chi V(g) = \mathrm{tr}(\rho V(g))$.

Definition 2.6. A character is *irreducible* if V is an irreducible representation.

Theorem 2.7. • If χ_1,χ_2 are two irreducible characters, then

$$<\chi_1,\chi_2>_G = \begin{cases} 1, \text{if} \chi_1 = \chi_2 \\ 0, \text{if} \chi_1 \neq \chi_2 \end{cases}$$

• (column orthogonality) If $g_1, g_2 \in G$, then

$$\sum_\chi \chi(\bar{g_1})\chi(g_2) = \begin{cases} \frac{|G|}{|C_G(g)|}, \text{if} g_1, g_2 \text{conjugate} \\ 0, \text{if} g_1, g_2 \text{notconjugate} \end{cases}$$

Definition 2.8. A *character table* is a 2-dimensional table whose rows are irreducible characters and columns are the conjugacy classes.

Here are some examples of the character tables constructed by using Theorem 2.8.

Example 2.9. We can construct the character table of S_4.

reps of S_4	e	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\chi triv$	1	1	1	1	1
χsgn	1	-1	1	1	-1
$\chi stand$	3	1	-1	0	-1
$\chi stand \cdot \chi sgn$	3	-1	-1	0	1
χv	2	0	2	-1	0

Example 2.10. Let D_4 be a group of symmetry operations (reflection and rotation) of a square. (i.e. $D_4 := \langle r,s \mid r^4 = s^2 = e, rs = sr^3 \rangle$, and note that D_4 has order 8). The character table of D_4 is:

reps of D_4	e	$\{r,r^3\}$	r2	$\{s,sr^2\}$	$\{sr,sr^3\}$
$\chi triv$	1	1	1	1	1
χr	1	1	1	-1	-1
χs	1	-1	1	1	-1
χsr	1	-1	1	-1	1
χv	2	0	-2	0	0

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Example 2.11. Let Q_8 be a quaternion group of order 8. Here is the character table:

reps of Q_8	1	$\{i, -i\}$	-1	$\{j, -j\}$	$\{k, -k\}$
$\chi triv$	1	1	1	1	1
χi	1	1	1	-1	-1
χj	1	-1	1	1	-1
χk	1	-1	1	-1	1
χw	2	0	-2	0	0

By combining Example 2.10 and 2.11 together, we see that dihedral group D_4 and quaternion group Q_8 have the exact same character table, but they are not the same group. In other words, they are not isomorphic to each other.

In order to prove that $D_4 \ncong Q_8$, we can list the elements of both groups explicitly:

$D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

Now, we need to compute the order of each element in each group:

1. e is the identity element in D_4, and 1 is the identity element in Q_8.

So they both have order equals to 1.

2. Now look at the element who has order equals to 2 in each group.

In D_4, r^2, s, sr, sr^2 and sr^3 all have order 2. So there are five elements have order 2.

In Q_8, only -1 has order equals to 2.

Therefore, there is no bijection between D_4 and Q_8, and this tells us that two groups can have the same character table without being isomorphic to each other.

3. Applications: Standard Young Tableaux (SYT)

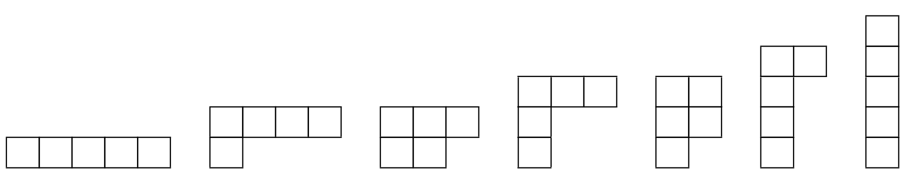
Throughout the section, we will discuss the applications in the Standard Young Tableaux (SYT).

First, we need to recall the definitions of the partitions and the SYT, the detailed definitions are in [Ful97].

Definition 3.1. (Young, A. (1900)). Suppose λ is a partition of n, then a *(Standard) Young Tableaux* is obtained by filling in each blocks with $1,2,...,n$ following those rules:

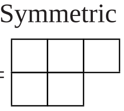
- every number occurs only once;
- the numbers increase across each row and down along each column.

Example 3.2. In S_5, we have 7 different conjugacy classes, and Theorem 2.3 tells us we have 7 partitions as well. Therefore, we can write out 7 distinct young diagrams of the representations in S_5.

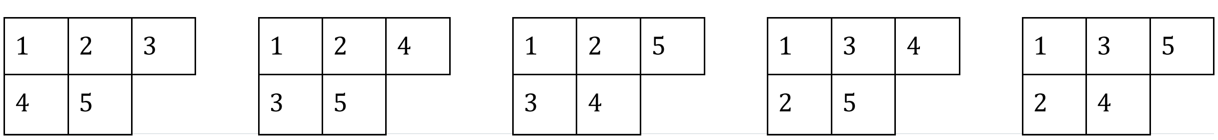


3.1 Hook length Formula

Prior to the advent of the *Hook length Formula*, the calculation of the total number of potential fillings for a SYT of shape λ was straightforward only when the tableau comprised a relatively small number of blocks. For example, in Symmetric Group S_5 with a shape

$\lambda =$ 

We can easily calculate the number of fillings by writing out the Young Diagrams explicitly as follows.



However, in a symmetric group Sn with a large number n, it is almost impossible to figure out the number of fillings straight away.

Therefore, the *Hook length Formula* was introduced to make such calculation within the bounds of possibility has played a consequential role in combinatorics.

Definition 3.3. (Hook length). Let λ be a shape of a SYT. For any cells $(i, j) \in \lambda$ located in i^{th} row and the j^{th} column, the *hook length* $h(i, j)$ is

$h(i, j)$ = (total number of row $- i$) + (total number of column $- j$) + 1.

The *Hook length Formula* introduced in [FRT54] says follows.

Theorem 3.4. (Frame-Robinson-Thrall). Let f^{λ} be the number of SYT of a shape λ, and $h(i, j)$ be the hook length of block $(i, j) \in \lambda$. Then

$$f^{\lambda} = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

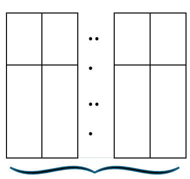
where $n = \lambda_1 + \lambda_2 + \cdots + \lambda k$.

Proof. There are numerous proofs for this theorem, the original proof constructed by the introducer themselves, however, is not intuitive enough. Since then, many mathematicians came up with some combinatorial and probabilistic proofs and they can be find in [GNW82], [Kra95] and [NPS97]. □

3.2 The (n, n) SYT

Here are some specific types of SYT which happens to have connection with some other famous applications in Algebraic Combinatorics.

First and foremost, let us consider the (n, n) SYT. Namely, the SYT which has 2 rows and each row has n columns.



n columns

Corollary 3.5. Let $n \in \mathbb{N}$. The number of fillings of the (n, n) SYT is

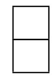
$$f^{(n,n)} = \frac{(2n)!}{n!\cdot(n+1)!}.$$

By Definition 3.3, we can easily see that the hook length of each blocks in the bottom column is 1,2, ... ,n (from right to left) since there is no blocks directly below each blocks and the number of the blocks increases by 1 from right to left sequentially.

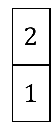
Similarly, the hook length of each blocks in the top column is 2,3, ... ,n, $n + 1$.

Proof. We shall use Theorem 3.4, and we complete the proof by using induction on n.

• *Base case*: $n = 1$.

Then the (1,1) diagram looks like 

By Definition 3.1, the hook length of the those two blocks are



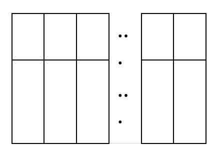
Then LHS $= f^{(1,1)} = \frac{2!}{2\times 1} = 1$ by Theorem 3.6, RHS $=\frac{(2\times 1)!}{1!\times 2!} = 1$.

Therefore, $P(1)$ holds.

• *Induction hypothesis*: we assume that $P(n)$ is true for all $n \in \mathbb{N}$.

• *Induction step:* Now consider $P(n + 1)$.

The diagram of $(n + 1, n + 1)$ SYT is:



n columns

which added one more column to the left of the (n, n) SYT.

By Definition 3.3, the hook length of the front column are $n + 2$ and $n + 1$ (from top to bottom respectively).

Then by Theorem 3.4,

$$\begin{aligned}
f^{(n+1,n+1)} &= \frac{(2(n+1))!}{\prod_{(i,j)\in(n+1,n+1)} h(i,j)} \\
&= \frac{(2n)!}{\prod_{(i,j)\in(n,n)} h(i,j)} \cdot \frac{(2n+1)(2n+2)}{h(1,n+1)\,h(2,n+1)} \\
&= f^{(n,n)} \cdot \frac{(2n+1)(2n+2)}{(n+1)(n+2)} \\
&= \frac{(2n)!}{n!(n+1)!} \cdot \frac{(2n+1)(2n+2)}{(n+1)(n+2)} \text{ (by induction hypothesis)} \\
&= \frac{(2n+2)!}{n!(n+1)(n+1)!(n+2)} \\
&= \frac{(2n+2)!}{(n+1)!(n+2)!} \\
&= \frac{(2(n+1))!}{(n+1)!((n+1)+1)!}
\end{aligned}$$

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Thus, we complete the proof.
Now, let us look at the Catalan Number *Cn*, which originally discovered in 1730s by Minggatu.
Definition 3.6. We define the n^{th} *Catalan number Cn* to be the number of triangulations of a regular polygon with (n + 2) vertices.
Note that we set $C_0 = 1$.
Theorem 3.7. The formula of the *Cn* is:

$$C_n = \frac{1}{n+1}(\text{\genfrac0}pt2nn)$$

Proof. This formula is proved by using the generating function which obtained by the recurrence relation

(Tex translation failed)

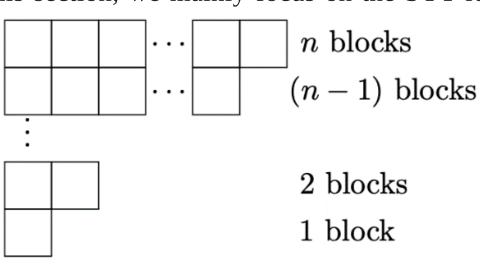
The detailed proof was written by Richard P. Stanley in his book called 'Catalan Numbers'. [Sta15] □
By combining Corollary 3.5 and Theorem 3.7, we can see that $f^{(n,n)} = Cn$.
We claim that there exists such bijection:

$$\left\{\begin{matrix}\text{numberoffillings} \\ \text{ofthe}(n,n)\text{SYT}\end{matrix}\right\} \leftrightarrow \left\{n^{\text{th}}\text{CatalanNumber}\right\}$$

To see this, suppose $\{a_k\}_{k=1}^{n} = \{a_1, a_2, \ldots, a_n\}$ and $\{b_j\}_{j=1}^{n} = \{b_1, b_2, \ldots b_n\}$ are two different sequences which satisfy $a_1 < a_2 < \cdots < an$ and $b_1 < b_2 < \cdots < bn$. The Catalan Number *Cn* is the number of ways to combine two sequences together to make a new sequence satisfying $\{a_k\}_{k=1}^{n} < \{b_j\}_{j=1}^{n}$. To be more specific, we can think of a mathematical model which has a connection with our daily life:
Suppose there are $2n$ people where they have different heights and we want to line them up into two rows with half of them in the first row and the other half in the second. The Catalan Number *Cn* is the number of arrangements that we can possibly have (up to reordering to make sure each row we have the tallest person on the left all the way to the shortest person on the very right) in order to satisfy the condition that people in the front row are all taller than people in the second row.

3.3 The Triangular Formed SYT

In this section, we mainly focus on the SYT looks like this:



In other words, the number of columns decrease sequentially by one from the top row which has n blocks to the bottom (1 block).
We now compute the number of ways to fill the triangular formed SYT by Theorem 3.6.

Corollary 3.8. Let $T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ be the total number of blocks in the SYT. The symmetric product group $STn = Sn \times Sn\text{-}_1 \times \cdots \times S_2 \times S_1$ can be represented as the diagram above.
The variety of ways to fill in such SYT is

$$\frac{\left(\frac{n\,(n+1)}{2}\right)!}{1^n \cdot 3^{n-1} \cdot 5^{n-2} \cdot \cdots \cdot (2n-3)^2 \cdot (2n-1)^1} \quad (3.1)$$

Proof. This follows by Theorem 3.4 □
Since the formula above cannot quickly and explicitly tell us the size of the irreducible representation of S_n, so we aim to find an asymptotic approximation of the formula.
We can easily approximate the numerator as follows by Stirling's formula:

$$\left(\frac{n(n+1)}{2}\right)! \sim \sqrt{2\pi \cdot \frac{n(n+1)}{2}} \left(\frac{n(n+1)}{2e}\right)^{\frac{n(n+1)}{2}} \quad (3.2)$$

$$= \sqrt{n(n+1)\pi} \left(\frac{n(n+1)}{2e}\right)^{\frac{n(n+1)}{2}} \quad (3.3)$$

We now concentrate on the denominator, and we may first rewrite it as a product of double factorials:

$$(2n-1)!! \cdot (2n-3)!! \cdots 3!! \cdot 1!! \quad (3.4)$$

We found an asymptotic approximation which derived by using Stirling's Formula in [OEI14, A006882]

$$n!! \sim \begin{cases} \sqrt{\pi} n^{\frac{n+1}{2}} e^{-\frac{n}{2}} \text{forevenn} \\ \sqrt{2} n^{\frac{n+1}{2}} e^{-\frac{n}{2}} \text{foroddn} \end{cases}$$

Therefore, the denominator becomes

$$\left(\sqrt{2} \cdot (2n-1)^n \cdot e^{-\frac{2n-1}{2}}\right) \cdot \left(\sqrt{2} \cdot (2n-3)^{n-1} \cdot e^{-\frac{2n-3}{2}}\right) \cdots \left(\sqrt{2} \cdot 1^1 \cdot e^{-\frac{1}{2}}\right) \quad (3.5)$$

$$= (\sqrt{2})^n \cdot e^{-\left(\frac{2n-1}{2} + \frac{2n-3}{2} + \cdots + \frac{1}{2}\right)} \cdot \prod_{k=1}^{n} (2k-1)^k \quad (3.6)$$

k=1

$$= (\sqrt{2})^n \cdot e^{-\frac{n^2}{2}} \cdot \frac{((2n-1)!!)^n}{(2n-3) \cdot (2n-5)^2 \cdots 3^{n-2} \cdot 1^{n-1}} \quad (3.7)$$

n

$$= (\sqrt{2})^n \cdot e^{-\frac{n^2}{2}} \cdot \frac{\left(\sqrt{2} \cdot (2n-1)^n \cdot e^{-\frac{2n-1}{2}}\right)}{\prod_{k=1}^{n-1} (2k-1)!!} \quad (3.8)$$

$$= 2^n \cdot e^{-\frac{3n^2-n}{2}} \cdot (2n-1)^{n^2} \cdot \frac{1}{\prod_{k=1}^{n-1} (2k-1)!!} \quad (3.9)$$

Now, combine equation 3.9 and 3.4 together,

$$\prod_{k=1}^{n-1}(2k-1)!! \sim 2^n \cdot e^{-\frac{3n^2-n}{2}} \cdot (2n-1)^{n^2} \cdot \frac{1}{(2n-3)!! \cdot (2n-5)!! \cdots 3!! \cdot 1!!}$$

By rearranging the equation, we can get:

$$(2n-1)!! \cdot \left(\prod_{k=1}^{n-1}(2k-1)!!\right)^2 \sim 2^n \cdot e^{-\frac{3n^2-n}{2}} \cdot (2n-1)^{n^2}$$

Then we multiply (2n − 1)!! on both sides to get:

$$\left(\prod_{k=1}^{n}(2k-1)!!\right)^2 \sim \left(2^n \cdot e^{-\frac{3n^2-n}{2}} \cdot (2n-1)^{n^2}\right)\left(\sqrt{2} \cdot (2n-1)^n \cdot e^{-\frac{2n-1}{2}}\right)$$
$$= 2^{n+\frac{1}{2}} \cdot (2n-1)^{n^2+n} \cdot e^{-\frac{3n^2+n-1}{2}}$$

Finally, by taking square on both sides, we get:

$$\prod_{k=1}^{n}(2k-1)!! \sim 2^{\frac{2n+1}{4}} \cdot (2n-1)^{\frac{n^2+n}{2}} \cdot e^{-\frac{3n^2+n-1}{4}} \quad (3.10)$$

which is the asymptotic approximation of the denominator.

Therefore, by putting equation 3.3 and equation 3.10 back into equation 3.1, we now have an asymptotic approximation to Corollary 3.8

$$\frac{\sqrt{n\,(n+1)\pi}\,\left(\frac{n\,(n+1)}{2e}\right)^{\frac{n\,(n+1)}{2}}}{2^{\frac{2n+1}{4}} \cdot (2n-1)^{\frac{n^2+n}{2}} \cdot e^{-\frac{3n^2+n-1}{4}}}$$

which can be simplified into

$$\frac{\sqrt{n\,(n+1)\pi}\,\left(n\,(n+1)\right)^{\frac{n\,(n+1)}{2}} \cdot e^{\frac{n^2-n-1}{4}}}{2^{\frac{2n^2+4n+1}{4}} \cdot (2n-1)^{\frac{n^2+n}{2}}}$$

In fact, the accurate calculation of the largest irreducible representations of symmetric group *Sn* for *n* up to 75 has already been done in [McK76]. When we look at the triangular numbers *Tn* where *Tn* = 1,3,6,10,... we can see that the highest order of irreducible representation of the symmetric group *STn* is exactly the number of ways to fill in the Triangular Formed SYT which we calculated in Corollary 3.8 by Hook Length Formula (Theorem 3.4). Therefore, we conclude that the greatest irreducible representation of a *STn* is the case where we represent it as a triangular formed SYT.

4. Conclusion

In this paper, we have used the SYT form to represent the symmetric group, with the goal is to analyze the dimension of it.

Nevertheless, in the case of the triangular formed SYT, the expression computed is not straightforward, so an asymptotic equation was evaluated as an approximation.

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