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Algorithmically determining whether a rational point is inside an open, constructive set or on the boundary of it is generally impossible

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Abstract:

We can determine whether this point is in the interior, boundary, or exterior of this set. In constructive mathematics, this is not always the case. The aim of this research is to demonstrate that it is generally impossible to algorithmically determine whether a point is in an open, constructive set or at its boundaries.

Keywords: Constructive Mathematics, Open Set, Rational Point, Unextendible Program

1. Introduction

Constructive mathematics is a subfield of mathematics which emphasizes constructive methods and proofs. In 1967, Bishop published the seminal monograph Foundations of Constructive Analysis [1], which laid a solid foundation for a program of mathematical research that thrived soon. In contrast to traditional mathematics, which concentrates on the existence of mathematical objects, constructive mathematics focuses on the explicit construction of mathematical objects. Another school of Constructive mathematics was developed by Markov, Shanin [2], and their followers in Russia. Kushner [2] wrote the text describing the Russian constructive mathematics approach. The difference between the Russian and the USA schools is that the Russian school believes in Markov's principle, which sometimes allows to include proofs by contradiction. [3]

Def 1.1 Decidable sets: let μ be some set of words in an alphabet *A*. We say that μ is algorithmically decidable if there is an algorithm that applies to every word in *A*, and it gives “*Yes*” when the word is exactly in μ and “*No*” otherwise.[4]

Def 1.2 Enumerable sets: A set μ is called algorithmically enumerable if we can construct an algorithm *U* over *A* such that $\forall$ n $\in \mathbb{N}$ and every word *P* in *A*.

If *U* (*n*) terminates then *U* (*n*) ∈ μ and if P ∈ μ then one can algorithmically find *i* such that U(*i*) terminates and U(*i*) = P. We say that *U* enumerates μ.

Def 1.3 Unextendible Program: A program that cannot be extended is an unextendible program that is partially defined fails to terminate for certain positive integer inputs and cannot be modified to work for all positive integer inputs. The theory of Computer Science holds the classical fact that unextendible programs exist; see A. Shen, and Vereshchagin N.K. [2].

This paper demonstrates that it is not always possible to determine algorithmically if a rational point is located on an open set's boundary, interior, or exterior. We choose a rational point r=1 and use an unextendible algorithm P (n) to construct an open set I (n) on the rational line.

Theorem: The interior or boundary of an algorithmically generated open set cannot always be determined by an algorithm.

General idea of the proof is as follows. We assume that it can always be algorithmically decided if the point is inside or on the boundary of an open set, we use such a decision program *Q (I, r)* to construct a total extension of the unextendible program P. This gives a contradiction. Hence, the theorem is proved.

2. Definitions

Def 2.1 Constructive Real Number (CRN): CRN is a combination of two computer programs, α(k) and β(k). α(k) is a computer-generated Cauchy sequence of rational numbers, and β(k) is the convergence regulator, i.e., the computer-generated sequence of positive integers, such that $\forall n \in \mathbb{N}$, |α(i) - α(j)| < 2^{-n} holds for $\forall i, j >$ β(n). The Standard Regulator is β(n) = n.

Def 2.2 Constructive function: An algorithm transforms every CRN into a CRN, which should take equivalent CRNs to equivalent CRNs. All usual functions, such as cosine, logarithm, etc., are constructive functions, Kushner [5].

Remark: In the work of Turing A. [4,6], Constructive Real Numbers first made their appearance in a slightly different form.

3. Notations

Symbols	Descriptions
E	An open constructive set
M	A fixed large number
I_n	The n-th open interval with rational endpoints
∂E	The boundary of the set
Int E	The interior of the set

4. Theorem

Algorithmic decisions about whether a rational point on the constructive real line is inside or outside an open set are generally impossible. If we prove the theorem assertion in this situation, it is possible to prove it in general for arbitrary constructive separable metric spaces, as the real line R is precisely the case in one dimension. See the discussion of constructive separable metric spaces later in this paper.

5. Proof

P(k) is an unextendible program that transforms positive integers to 0 or 1. Given a fixed rational number 1, we can consider k=1 at first. Define a sequence of open intervals

$I_n(k) = \begin{cases} (1+2^{-n}, M), n < N \\ (1-2^{-N}, M), n \geqslant N \end{cases}$ If the program terminates on N-th step

We get the open set I (1), which I (1) is the union of In (1). Assume there is a program Q that, given an open set and a rational point, produces 1 if the point is in the interior of the open set and produces 0 if the point is in the boundary of the open set. Apply Q to the pair I (k) and the rational number 1. This program prints 0 exactly if P (k) never terminates and it prints 1 exactly if P (1) eventually terminates.

Similarly, apply Q to the pairs I(k), k=2, 3 ... and the rational number $r = 1$. Now we see that we have a decision algorithm that tells if P(k) will eventually terminate, but the domain of P is undecidable. Thus, we have a contra-

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diction, and the program Q cannot exist.

Remark: the open set in the theorem proof consists of infinitely many intervals so to avoid misunderstanding we apply the decision algorithm Q to the computer code that generates this infinite union rather than to the union itself.

Def 5.1 Separable constructive metric topological space: We have a collection of algorithmically given points. For any pair of points, there is a program that gives the distance between them, which is CRN. The topological space we discussed above should have an enumerable basis.

Since the definition of separable constructive metric topological space and theorem holds for the real line, it will hold for such spaces as well.

6. Conclusion

The result we provide is that it is generally impossible to determine whether a rational point is inside or outside an open set based on constructive mathematics insight.

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