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The Non-universality of an Algorithm to decide if a point is in the exterior or on the boundary of a closed constructive set

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Abstract:
It seems we can find if a point is on the boundary or in the exterior of a closed constructive set. However, this is not true in constructive math. This paper investigates the algorithmic decidability of determining the position of rational points relative to closed sets given as the intersection of closed intervals with rational endpoints.

Keywords: Constructive mathematics, algorithm, rational point, close set.

1. Introduction

First appeared in the late 19th century, the early ideas of constructive mathematics are quite different from classical mathematics in essence. In constructive mathematics [1], proofs of existence solely are not always accepted; such a thing must be constructible so that we acknowledge it really exists. This field of study was promoted by Brouwer's early ideas of Intuitionism. According to his philosophical proposition, the universal validity of contradiction proofs for existence proofs was unwarranted. Born in 20 Jul. 1922, the Russian mathematician Andrei Andreyevich Markov contributed mainly to probability, number theory and mathematical analysis. He had come up with a brilliant idea that is crucial to many subjects, including constructive mathematics, called the Principle of Constructive Choice, or Markov's principle, giving the following formula(mathematical statement):

$$\neg\neg !u(x) \supset !u(x)$$

where the symbol "!" is the sign for the statement that the algorithm u terminates on the input x. It can be read as: If it is not the case that there does not exist a natural number x such that u(x) holds, then there exists a x for u(x) to hold. It sometimes allows us to argue by contradiction. However, this principle is not accepted by all constructive mathematicians since it does not tell you how long the program will take to find such a x. Around the 1950s, with the works of Markov, Shanin, Kushner, and so on, the USSR started a systematic study of constructive analysis and made obvious progress [2].

Besides A. A. Markov, there's also an important person we have to know. Errett Albert Bishop (July 14, 1928 – April 14, 1983) was an American mathematician known for his work on analysis. His work [3] shows that a constructive treatment of analysis is feasible and lays a foundation which contributes to future studies. However, Bishop followers do not accept the constructive choice principle which Markov produced.

Many people misunderstand constructive mathematics as classical mathematics with removed law of excluded middle and/or axiom of choice. In reality it intends to give a more strict interpretation of proof of existence. For constructive mathematicians the words that an object exists mean that it can be constructed as an output of a computer program. Modern constructive mathematics is well integrated with computer science, realizing mathematical construction processes by algorithms.

2. Definition

Def 2.1 Constructive: It means that when a mathematical object is asserted to exist, an explicit example is given: a constructive existence proof demonstrates the existence of a mathematical object by outlining a method of finding ("constructing") such an object [4].

Def 2.2 Closed Set: In geometry, topology, and related branches of mathematics, a set is closed if and only if it coincides with its closure. In other words, a closed set can be defined as a set that contains all its boundary points [5].

Def 2.3 Regulator: A convergence regulator for a computer-generated sequence of rational numbers $\alpha(i)$ is an CSNN (Constructive Sequence of Natural Numbers) such that for $\forall n \in \mathbb{N}$ and $i, j \geqslant \beta(n)$, we have $|\alpha(i) - \alpha(j)| < 2^{-n}$. A regulator is standard if $\beta(n) = n$.

Def 2.4 Un-extendible: An un-extendible program, in the realm of theoretical computer science, refers to a partially defined computer program that encounters non-termination for a subset of positive integer inputs and, fundamentally, cannot be transformed or extended into a program that ensures termination and proper functionality across all positive integer inputs. A well-established fact within the field is the existence of such unextendible programs, highlighting the inherent complexity and limitations in designing universally applicable computational algorithms. see Shen, A. and Vereshchagin N.K [6]

Def 2.5 CRN(constructive real numbers) [7,8]*:* Combination of two computer programs α(n) and ß(n), in which α(k) is a sequence of rational numbers and ß(k) is a sequence of positive integers, such that for ∀ n ∈ ℕ, |α(x)-ß(y)| < 2^(-n) holds for ∀ x, y > ß(n).

Def 2.6 Constructive Metric Spaces: A list { S,ρ }, where S is a set of constructive words and ρ is an algorithm converting any pair of elements of S into a CRN, is called a constructive metric space [9,10].

Def 2.7 Separable Constructive Metric Space: Let M = { S,ρ } be a constructive metric space. M is called separable if there exists algorithms α,δ such that α is a sequence of points of M for any xϵM and for any n, $\delta(X*n)$ is a natural number where $\rho(\alpha(\delta(X*n)),X) < 2^{-n}$.[9,10]

3. Theorem

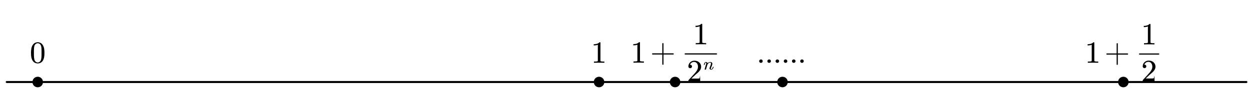
It is generally impossible to algorithmically decide whether a rational point is in the exterior of an algorithmically given closed constructive set or is on the boundary of it. In response to the above question, we need to remark that the example we construct happens on a real line , which is in the 1st dimensional. The example Q(k,1) we use is a specific program to test whether 1 is in the exterior or on the boundary of the constructive closed set. S_k

4. Proof

Use 1 as the point. Take an un-extendible program P(k). Then, we algorithmically define a sequence of closed intervals: $S_{n(k)}$ as follows. If program P is still running on input k on step n then we put $S_{n(k)} = \left[0, 1 + \frac{1}{2^n}\right]$, so that if P(k) never terminates then we generate a closed set:

$$S_k = \bigcap_{n=1}^{+\infty} S_{n(k)}$$

$$S_k = [0, 1]$$



If the program terminates at N-th stage for a fixed N≤n

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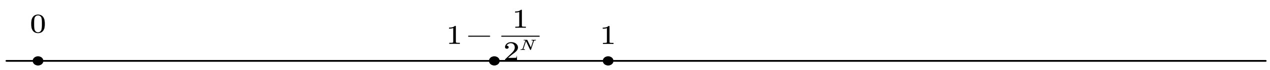
on input k, we define $S_{n(k)} = \left[0,1-\frac{1}{2^N}\right]$ for all n≥N. In both cases we put $S_k = \bigcap_{n=1}^{+\infty} S_{n(k)}$. So that if P(k) will never stop working then S_k = [0,1] and it has the rational point 1 on the boundary of it. If P(k) stops working eventually then then S_k = [0, 1- $\frac{1}{2^N}$] and 1 is in the exterior of this interval. Here N is the step number when the program P(k) stopped working.



Now, we argue by contradiction and assume there is a program Q (C, r) that, given a closed set C and a rational point, r can always decide if r is in the exterior of C or on the boundary. Apply this program Q to the pair S_k and r=1 to get the decision algorithm for a domain of P. This gives a contradiction since the domain of P is undecidable, but Q exactly decides if P(k) will terminate eventually or not.

5. Remark

In reality, the program Q takes as an input the algorithm that generates the closed set as the intersection of infinitely many closed sets rather than the input of infinitely many closed sets. The reason why we use powers of 2 in the construction of S_k is that we want to get an interval with CRN end points.

6. Conclusion

To sum up, we yield the eventual consequence that it is generally impossible to algorithmically determine whether a point is in the exterior or on the boundary of a closed constructive set. The method we use is to contradict the decidability of whether a point is on the boundary or exterior of a set with the fact that the domain we used to construct this set is undecidable. Since the constructive real line is an example of a constructive separable metric space, the same conclusion holds for the points in the constructive separable metric spaces.

Appendix

Table 1. Notations

Symbols	Meaning
$S_{n(k)}$	A sequence of closed intervals
S_k	A constructive closed set created as the intersection of $S_{n(k)}$
$P(k)$	An un-extendible program with input k
$Q(k)$	A conjecturally existing program that can always determine the position of rational point r in S_k, which is whether r is in the exterior of S_k or in the boundary of it

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