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Definition, stability analysis and numerical simulation of FitzHugn-Nagumo model

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Abstract:

In this paper, the definition, nonlinear mechanism and numerical analysis of FitzHugn-Nagumo model are introduced. FitzHugn-Nagumo model is used to describe the action potential and recovery process of neurons. In this paper, the definition of FitzHugn-Nagumo model [1] is first given. FitzHugn-Nagumo model is used to describe the action potential and recovery process of neurons. In this paper, the nonlinear mechanism of the model is considered, the fixed point of the model is calculated, the linear stability of the model is analyzed, the characteristic value of the model and the system behavior are analyzed, and the formation mechanism of Hopf bifurcation, critical conditions and limit cycles are discussed. The Euler method, the most commonly used numerical algorithm, is used to visually discuss the system behavior, and the rationality and validity of FitzHugn-Nagumo model are verified, which provides an important reference for the study of neuronal dynamics.

Keywords:

FitzHugn-Nagumo Model, Neuronal Dynamics, Numerical Algorithm, Euler Method, Stability analysis, Fixed point, Limit cycle, Periodic oscillation

Introduction

FitzHugn-Nagumo model is an important model of neuronal dynamics, which was proposed by FitzHugh in 1961. FitzHugn-Nagumo model is a two-dimensional simplification of Hodgkin-Huxley model [2]. The Hodgkin-Huxley model describes the mechanism of neuron action potential in detail, but because of its complexity, it is difficult to perform mathematical analysis and numerical calculation. FitzHugn-Nagumo model successfully simplifies the model by introducing two variables, namely membrane potential and recovery variable, while retaining the basic dynamic characteristics of neuron firing.

In recent years, the FitzHugn-Nagumo model has been widely used in the fields of neuroscience, biophysics and nonlinear dynamics. In this paper, the dynamic behavior of FitzHugn-Nagumo model is verified by nonlinear dynamic analysis and numerical simulation.

This paper firstly defines the mathematical form of FitzHugn-Nagumo model, time scale separation, analysis of the model's fixed point, linear stability analysis. Then, the Euler method, a commonly used numerical algorithm, is introduced, and the dynamic behavior of FitzHugn-Nagumo model is analyzed visually. Finally, the dynamic properties of FitzHugn-Nagumo model are analyzed and discussed in combination with the nonlinear theoretical analysis and numerical simulation results, and the rationality and validity of FitzHugn-Nagumo model are verified, which provides enough reference for future research.

I.Model definition

This paper mainly studies the FitzHugn-Nagumo model of two-dimensional systems. [3][4]

$$\epsilon \frac{du}{dt} = u - \frac{u^3}{3} - v \,(1)$$

$$\frac{dv}{dt} = u + a \,(2)$$

The FitzHugn-Nagumo model is a two-dimensional simplification of the Hodgkin-Huxley model [5], which is generated by spikes in the giant axon of squid. Here u is the membrane potential, which represents the electrical activity of neurons and is the recovery variable, describing the activation state of full ion channels in neurons. The recovery variable v regulates the change of the membrane potential by coupling with the membrane potential, thus affecting the excitation and repolarization process of neurons. The recovery variable is usually related to the activation of full ion channels, and the opening and closing speed of these channels is slow. Therefore, the rate of change of recovery variables is also slower. A small amount ϵ ($\epsilon \ll 1$), representing a time scale, means that the change in the recovery variable is much slower relative to the change in the membrane potential. When the membrane potential rises, the recovery variable gradually increases, causing the membrane potential to begin to decline, thus simulating the repolarization process of neurons, and the recovery variable helps the system to return to a stable equilibrium state by adjusting the membrane potential. a is a control parameter that determines the dynamic characteristics of the system.

II.Time scale separation

Because $\epsilon \ll 1$, the change of$u$ in equation (1) is very fast, while the change of v in equation (2) is relatively slow. This separation of time scales allows us to divide the motion of the system into fast and slow parts:

Fast motion: v remains almost constant for a short period of time, so the dynamics of u are mainly determined by equation (1), that is, x will quickly trend towards its instantaneous equilibrium point $x \approx \frac{x^3}{3} + y$.

Slow motion: Over a longer period of time, y slow motion follows the average $y \approx x - \frac{x^3}{3}$.

III. Linear stability analysis

In this paper, the effects of parameter ranges on the stability of the FitzHugn-Nagumo model and the existence of periodic oscillations are discussed.

1.Solution of fixed point

The fixed point of equation (1)(2) is determined by the following conditions: $x - \frac{x^3}{3} - y = 0 and$ $x + a = 0$, Therefore, $x = -a, y = -a + \frac{a^3}{3}$,the fixed point is: $x = -a, y = -a + \frac{a^3}{3}$.

2. stability analysis

In order to analyze the stability of the fixed point, we calculate the Jacobian matrix as: : $J = \begin{pmatrix} 1-x^2 & -1 \\ 1 & 0 \end{pmatrix}$, at the fixed point $(x, y) = (-a, -a + \frac{a^3}{3})$, the Jacobian matrix becomes: $(x, y) = (-a, -a + \frac{a^3}{3})$, λ determined by the following characteristic equation: $det(J - \lambda I) = 0$, $\lambda_{1,2} = \frac{-(a^2-1) \pm \sqrt{(a^2-1)^2 - 4}}{2}$, expand the determinant to obtain the quadratic equation: $\lambda^2 + (a^2-1)\lambda + 1 = 0$ Using the root-finding formula, the eigenvalues are: $\lambda_{1,2} = \frac{-(a^2-1) \pm \sqrt{(a^2-1)^2 - 4}}{2}$.

3. Properties of eigenvalues and system behavior

According to the discriminant of the characteristic equation $(a^2-1)^2 - 4$, we can divide the eigenvalues into three categories:

a. $|a| > 1$,Real eigenvalues: $(a^2-1)^2 - 4 > 0$,At that time, the eigenvalues were two real numbers. If $(a^2-1) > 0$, then both λ_1 and λ_2 are negative real numbers, indicating that the fixed point is a stable node. This means that all trajectories converge to this fixed point, the system is stable, any small perturbation will bring the system back to the fixed point, and the system will not exhibit periodic or oscillatory behavior. If $a^2 - 1 < 0$, λ_1 is a positive real number, λ_2 is a negative real number, indicating that the

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fixed point is a saddle point. In this case, the system has a stable direction and an unstable direction, and the system will move towards a fixed point in some directions but away from it in others. This behavior is similar to the movement of an object on a saddle.

b. At that time $|a|<1$, conjugate complex eigenvalues: At that time $\left(a^2-1\right)^2-4<0$, the eigenvalues were a pair of conjugate complex numbers.The real part of the eigenvalue: $Re(\lambda)=\frac{1-a^2}{2}$ Since $\left(1-a^2\right)>0$,the real part of the eigenvalue is positive, it means that the fixed point is the focus of instability. The imaginary part of the eigenvalue: $Im(\lambda)\neq 0$, indicates that the system will have oscillation behavior, and the system will have a limit cycle, that is, the system will cycle periodically in the phase space, which is manifested as a pulse-like behavior.

c. At that time $|a|=1$, the eigenvalues of pure imaginary numbers: $\left(a^2-1\right)^2=0$ at that time, the eigenvalues were pure imaginary numbers. The form of the eigenvalue: $\lambda_{1,2}=\pm i$,, indicates that the system is in a critical state, and the fixed point is neither unstable nor stable, but in a central state.

IV. FitzHugn-Nagumo model numerical analysis algorithm

1.Euler method

Ordinary differential equation problems[6][7][8] will bring the derivative discretization approximation into the equation to obtain an iterative formula:

$$u=u+\frac{u-\frac{u^3}{3}-v}{\epsilon}*h \quad (3)$$

$$v=v+\left(u+a\right)*h \quad (4)$$

2.simulation programming language

```
program main
implicit none
real*8 :: u,v,utem,vtem,epsilon=0.01,a=1.0,h=0.001
real*8 :: D=0.0
real*8 :: t
integer*4 :: it
open(10,file='t_u_v.txt')
u=0.0;v=0.0;t=0;it=0
do while(t<100)
utem=u+(u-u**3/3.0-v)/epsilon*h
vtem=v+(u+a)*h
u=utem;v=vtem
it=it+1
t=h*it
if(mod(it,10)==0) write(10,*)t,u,v
enddo
close(10)
call system('gnuplot plty.txt')
end program main
```

3. Drawing pl script

```
set term qt size 800,600
set xlabel 'u'
set ylabel 'v'
set title 'v vs u'
plot 't_u_v.txt' using 2:3 with lines title 'v-u' lc"blue"
pause -1
```

4. simulated result

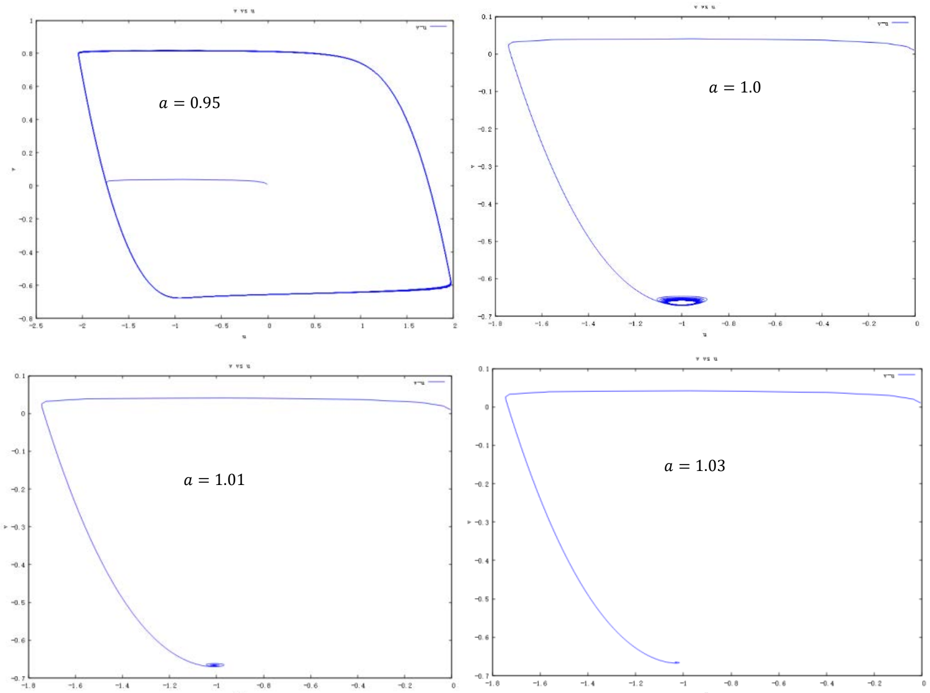

Figure 1. The initial value is (0,0), and the system takes the phase diagram of 0.95,1.0,1.01, and 1.03 respectively

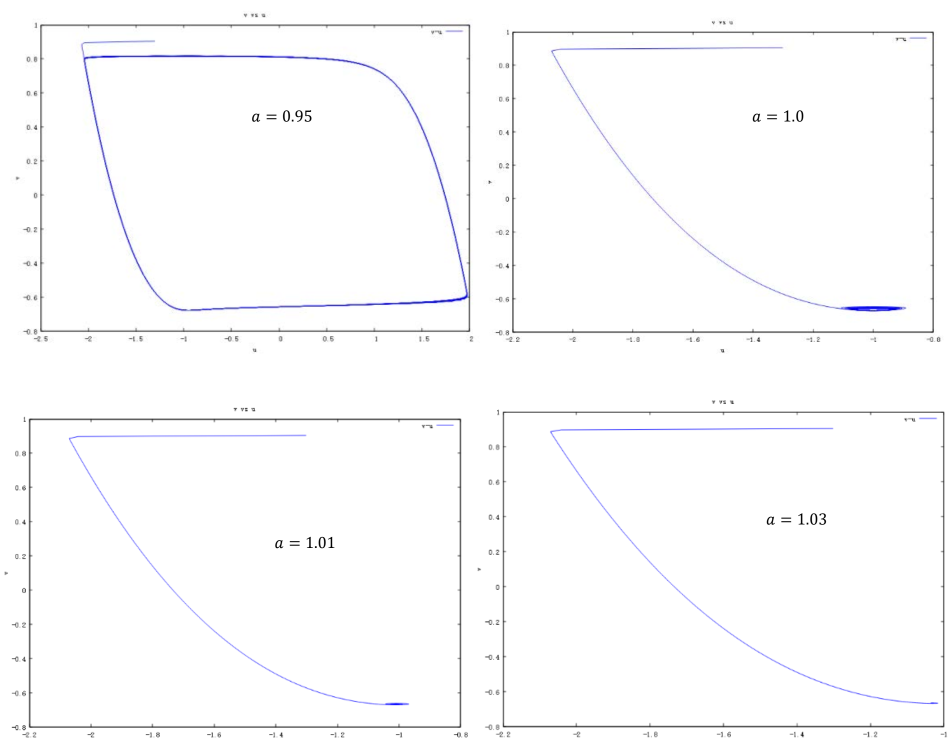

Figure 2. The initial value is (0,0.9), and the system takes the phase diagram of 0.95,1.0,1.01, and 1.03 respectively

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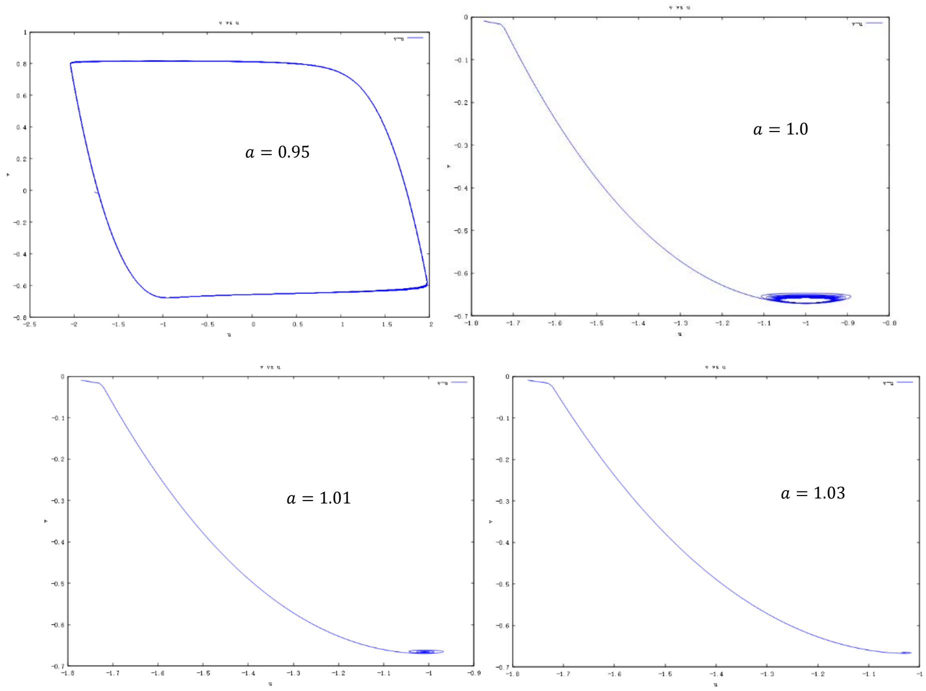

Figure 3. The initial value is (-2.4,0), and the system takes the phase diagram of 0.95,1.0,1.01, and 1.03 respectively

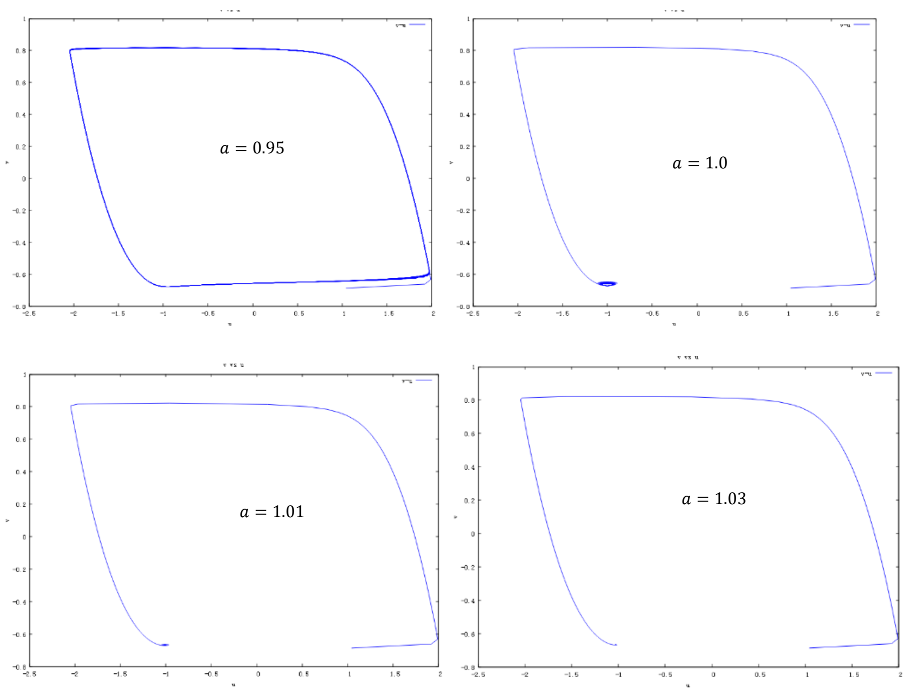

Figure 4. The initial value is (0,-0.7), and the system takes the phase diagram of 0.95,1.0,1.01, and 1.03 respectively

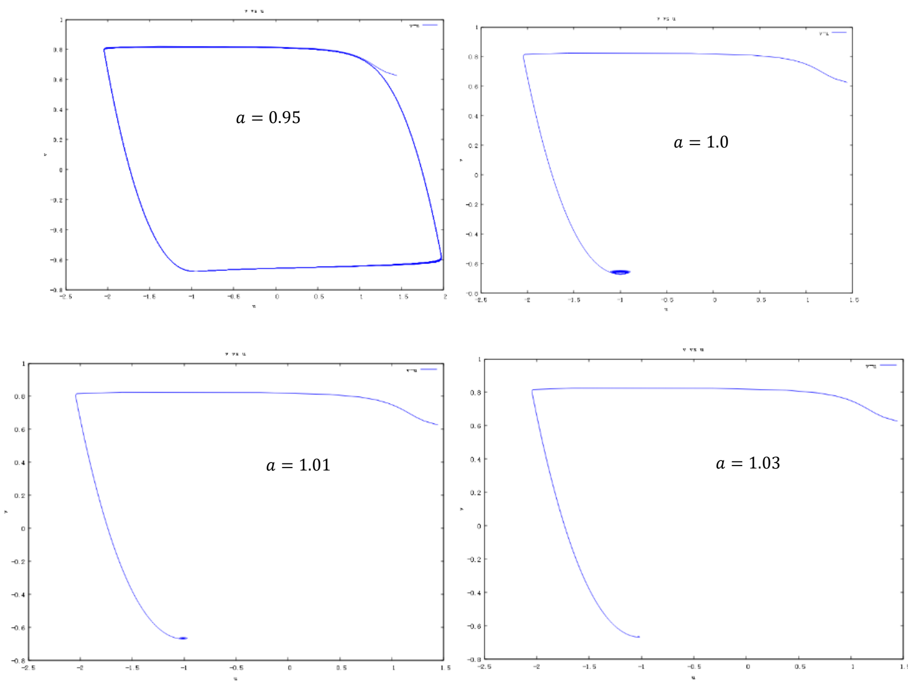

Figure 5. The initial value is (1.9,0.6), and the system takes the phase diagram of 0.95,1.0,1.01, and 1.03 respectively

V.Summary and discussion

It can be observed in Figure 1 that in the FitzHugn-Nagumo model, when the system starts from the initial value $(0,0)$, $|a|<1$ the system shows the limit cycle of periodic oscillation. $|a|=1$,When the system is in the center state. When the system is stable, the system will converge to a fixed point $(-1.02,0.67)$. Graph2, graph3, graph4, graph5with initial values $(0,0.9),(-2.4,0),(0,-0.7),(1.9,0.6)$ above, to the left and below the limit cycle, on the right, when the system starts from initial values, $|a|<1$ the system shows the limit cycle of periodic oscillation. $|a|=1$ When the system is in the center state. $|a|>1$ When the system is stable, the system will converge to a fixed point $(-1.02,0.67)$.

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