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The Application of Cauchy Integral Formula to Functions

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Abstract:

The Cauchy integral formula is a fundamental and critical formula in the complex function, and it has a variety of applications in many aspects such as physics, engineering, and mathematics. This article focuses on the applications in the functions and two main theorems in the Cauchy integral formula. It explores the formula and provides corresponding proofs of the formula. To this end, the aim of the article is to use the Cauchy integral formula to cope with some difficult and complex integrals and functions. As a result, this article denotes to use the formula to simplify the complex and difficult problems and provides a practical and fast method. Hence, the article lists some examples to demonstrate the functions' use of the Cauchy integral formula. At last, the article may serve as a reference for relevant research and survey in the field of the complex functions and provide insights into the Cauchy integral formula's uses.

Keywords: Cauchy integral formula; Analytic function; Singular points.

1. Introduction

Complex variable function was born in eighteenth century, mainly established by Euler, D'Alembert, Laplace and other mathematicians. By nineteenth Century, due to the effort by Cauchy, Riemann and other mathematicians, the theory of complex function has been fully progressed and become a very popular new branch of mathematics [1,2]. Today the complex function has a history of more than and 300 years, extensively utilized in numerous natural scientific domains, including electrical engineering, quantum information, signal processing, automatic control theory, fluid mechanics, elastic mechanics, aerodynamics, and theoretical physics, as well as quantum computing. And Cauchy integral formula is a key, it engraved another definition of the analytical function; and it was studied meaningfully, allowing the theory of analytical functions to be separated from real functions. The value on the border of the simple closed curve C represents the value of the solution function $f(z)$ at any position inside the curve using the Cauchy integral formula [3]. This is another feature of the analysis function.

In recent years, there are more and more research and surveys about Cauchy integral formula, they covered many aspects of the uses of the Cauchy integral formula such as the application in the higher derivative [4], in the real function and in the Cauchy residue theorem [5]. At the same time, they have achieved many excellent results. For example, Cauchy integral

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formula can be used to deduce the Cauchy integral higher derivative formula. And as a vital tool in the research of the resonance function formula, Cauchy integral higher derivative formula describes an important feature of the analytic function -- the existence of arbitrary conductors of the solution function within its area of resolution, and the result that any conductors can be written as points along the boundary curves of the region. Moreover, the Cauchy integral higher derivative formula is a method to prove many significant results like the Lewisville theorem, the Morella theory, and Cauchy inequality.

Additionally, the theorem is a crucial importance in the complex number and the calculus. On the one hand, this formula introduces the definition and function of the analytic function. It is a convenient method to solve and calculate the cumbersome function and calculus. And the condition of using this method is just judge the function whether is derivable in its range or not. If it is derivable, it only needs to use the formula to calculate the value of the calculus directly. If not, it needs to find the points which the function is not derivable, and it can lead to a new definition. It is residue, and there is a corresponding theorem of the residues—Cauchy residue theorem—another vital theorem in the field of complex number and calculus. It just needs to use $2\pi i$ multiple the residue so that calculate the integral. Overall, this formula simplifies the process of calculating integral and provide a clear way—using analytic map and finding the residues. Also, it paves the way to Cauchy residue theorem and provides some new definitions in the complex number field such as residue, singularity, and contours. On the other hand, the Cauchy integral theorem has a profound effect on complex numbers. The complex number is an abstract and tough to understand definition. This is because it has two sections—the real part and the imaginary part, which results that it is difficult to calculate the values and the integrals of the complex number though the traditional methods. Hence, the theorem is an ideal choice, it just needs to judge the function whether is analytic on its contour to calculate its integral. So, it isn't affected by the imaginary part. Totally, the Cauchy integral formula is significant because it fosters the development of the complex number and the calculus.

This article introduces the application of the theorem in the analytic functions. The formula is named after Augustin Louis Cauchy, a mathematician, physicist, and astronomer known for his work on calculus. This article introduces two theorems in the formula. They are Cauchy integral theorem and Cauchy-Gour sat theorem. These two theorems are widely used in functions of complex variables and calculus. In the Section 2, the article introduces the concepts and contents of these two theorems. Also, it offers the verification of these two theorems. In the Section 3, a variety of examples and applications of the formula in the analytic function are presented, while the Section 4 concludes this article.

2. Method

2.1 Cauchy Integral Theorem

Let f, in the positive sense, be analytic everywhere belongs to on a simple closed contour C. Any point inside of C, denoted by z_0, then the value of function $f(z)$ at z_0 is equal to

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}dz \tag{1}$$

The Cauchy integral theorem formula is represented by equation (1). It indicates that the values of f on C entirely determine the values of f interior to C if a function f is to be analytic both within and on a simple closed contour. When the Cauchy integral formula is written as:

$$\oint \frac{f(z)}{z - z_0} = 2\pi i f(z_0) \tag{2}$$

It is applicable to the evaluation of certain integrals along elementary closed contours

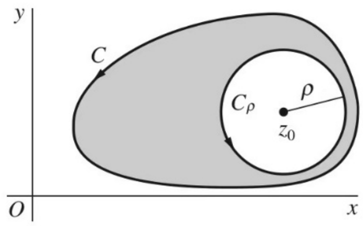

Figure 1. Illustration of the contour C and the pole z_0 [6]

The author starts the demonstration of the theorem by defining $C\rho$ as a positively oriented circle with formula $|z - z_0| = \rho$, where ρ is small enough for $C\rho$ to be interior to C. The quotient $f(z)/(z - z_0)$ exhibits analytic behavior on and between contours $C\rho$ and C, suggesting that the use of route deformation theory can provide that

$$\oint \frac{f(z)}{z - z_0}dz = \oint_{C\rho} \frac{f(z)}{z - z_0}dz \tag{3}$$

This enables us to write

$$\oint \frac{f(z)}{z - z_0}dz - f(z_0)\oint_{C\rho} \frac{1}{z - z_0}dz = \oint_{C\rho} \frac{f(z) - f(z_0)}{z - z_0}dz \tag{4}$$

However, according to this below equation

$$\oint_{C\rho} \frac{1}{z-z_0} dz = 2\pi i$$

Later, the equation (3) can be expressed

$$\oint \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) = \oint_{C\rho} \frac{f(z)-f(z_0)}{z-z_0} dz \quad (5)$$

Currently, f ensures that for each positive integer ϵ, regardless of its value, there exists a corresponding positive number δ that is analytic and continuous at z_0.

$$|f(z)-f(z_0)| < \epsilon whenever |z-z_0| < \delta \quad (6)$$

In the second inequality, let the radius ρ of the circle $C\rho$ be smaller than the number δ. When z is on $C\rho$, one can deduce that the first of the five inequalities are true. This indicates that and provides upper limits on the integrals' moduli surrounding outlines.

$$\left| \oint_{C\rho} \frac{f(z)-f(z_0)}{z-z_0} dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon \quad (7)$$

Next, focusing on Eq. (4), then it is arrived that

$$\left| \oint \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| < 2\pi\epsilon \quad (8)$$

Given that the nonnegative constant on the left side of this inequality is smaller than the arbitrary small positive number, it follows that $\oint \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) = 0$. As a result, the theorem is proved, and equation (1) is true. When the theorem is written as

$$\oint \frac{f(z)}{z-z_0} = 2\pi i f(z_0)$$

It can be used to evaluate certain integrals along closed, basic shapes.

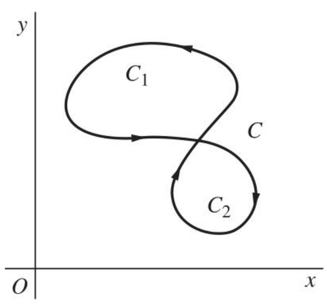

Fig. 2 Illustration of the contours C and C_1 and C_2

2.2 Cauchy-Gour sat theorem

If there is a function f, all points on the contour of the function can be derivable then

$$\oint f(z) = 0 \quad (9)$$

Additionally, there is a closed contour C and it has a finite number of crossings, the Cauchy-Gour don theorem can be used again. Here, C is composed of a finite number of closed, simple shapes. This may be seen in the graph below, where C is composed of the two simple closed contours C_1 and C_2. The values of the integrals surrounding C_1 and C_2, independent of their orientation, are zero. Then, on the contour C, every point on the function f follows that

$$\int_c f(z) dz = \int_{c_1} f(z) dz + \int_{c_2} f(z) dz = 0 \quad (10)$$

Proof. Let C represent a positive (counterclockwise) simple closed contour $z = z(t)(a \leqslant t < b)$, and assume that f is analytic at all points on and inside of C.The integral of $f(z)$ can be deduced to this form

$$\oint f(z) dz = \int_a^b f[z(t)] z'(t) dt \quad (11)$$

If $f(z)$ and $z(t)$ satisfy $f(z) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$, the integrand $f[z(t)]z'(t)$ in the above equation is equal to the multiple of the functions $u[x(t), y(t)] + iv[x(t), y(t)]$ and $x'(t) + iy'(t)$. Regarding the actual variable t, it is found that

$$\oint f(z) dz = \int_a^b (ux' - vy') dt + i \int_a^b (vx' + uy') dt \quad (12)$$

Thus, the former function can be expressed by two integrals including the variables

$$\oint f(z) dz = \oint udx - vdy + i \oint vdx + udy \quad (13)$$

Note that Eq. (13) can be produced formally by extending the product of the binomials $u + ivanddx + idy$, correspond to the positions of $f(z)$ and dz on the left. Of course, Eq. (3) also holds when $f[z(t)]$ is merely a continuous function on its contour C and when C is any contour, a simple closed one.

The author now revisits a calculus result that allows the integrals written in Eq. (13) to be written as double integrals. Assume for the moment that there are two real-valued functions, $P(x,y)$ and $Q(x,y)$, they are both analytic and continuous on their contour R, which is made up of all points on the simple closed contour and within it. C. According to the Green Theorem [6],

$$\oint Pdx + Qdy = \int \oint (Q_x - P_y) dA \quad (14)$$

Since f is analytic on R, it is now continuous there. The functions u and v are therefore also continuous on R

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. Similarly, the first-order derivatives of u and v are also continuous on R if the derivative f' of f is. Due to Green's theorem, Eq. (13) may thus be expressed as

$$\oint f(z)dz = \int\oint(-v_x - u_y)dA + i\int\oint(u_x - v_y)dA \quad (15)$$

However, according to the Cauchy–Riemann equations $ux = vy$ and $uy = -vx$, the integrals of these items presented in the upper equation are equal to zero across the whole range of R.Therefore, if f is continuous and analytic in R,

$$\oint f(z)dz = 0 \quad (16)$$

It is worthwhile to note that Cauchy arrived to this conclusion around the beginning of the 1800s.

Keep in mind that the orientation of C is irrelevant once it has been determined that the value of this integral is zero. That example, if C is included in the statement (5), then clockwise way ever since $\oint f(z)dz = -\oint f(z)dz = 0$.

Gour sat was the first to show that the continuity condition on f' was missing. Its elimination is important because it will allow us to show, for example, that an analytic function f' is derivative f is analytic without needing that f' be continuous. which subsequently takes place. Cauchy's result has been revised and is now presented. People refer to this as the Cauchy-Gour sat theorem [7].

3. Examples

3.1 Example 1

According to the value of integral

$$I_1 = \oint \frac{1}{z+3}dz \quad (17)$$

(C:$|z|=1$), the aim is to calculate the integral [8]

$$I_2 = \oint_0^p \frac{1+3cosx}{10+6cosx}. \quad (18)$$

To begin with, this paper shall find this integral $I_1 = \oint \frac{1}{z+3}dz$. The integral is a fraction, and its denominator is $z+3$. When z equal to minus three, it can get that $z+3=0$,the denominator is equal to zero, and this fraction does not exist. So, the function is not analytic when z equal to minus three. However, the contour of z is $|z|=1$, $z=-3$ doesn't in this contour. So, the denominator isn't zero on the contour, and it can be analytic on its contour. This is because the integral $\oint \frac{1}{z+3}dz$ is analytic on its contour, it satisfies the condition of the Cauchy-Gour sat theorem. Thus, the value of integral $\oint \frac{1}{z+3}dz = 0$.

Then, substitution $z = cosx + isinx, (-p \leqslant x \leqslant p)$, and the integral $\oint \frac{1}{z+3}dz$ can be rewrite as

$$I = \oint \frac{dz}{z+3} = \oint_{-p}^{p} \frac{-sinx + icosx}{cosx + isinx + 3}dx, \quad (19)$$

and it multiplies the expression by $cosx + 3 - isinx$ and simplifies, the integral converts that

$$I = \int_{-\pi}^{\pi} \frac{-3sin\theta}{10+6sin\theta}d\theta + i\int_{-\pi}^{\pi} \frac{1+3cos\theta}{10+6cos\theta}d\theta \quad (20)$$

And it needs to calculate the second item, the range of second item is from $-\pi$ to π . In advance, simplify the range of the integral, it can be rewritten

$$I = 2i\int_0^{\pi} \frac{1+3cos\theta}{10+6cos\theta}d\theta \quad (21)$$

Because it is analytic on its contour, so the integral is equal to zero. As a result, the integral that the topic requires to calculate is equal to zero divided by $2i$, it also equals to zero. [8]

$$I = 2i\int_0^{\pi} \frac{1+3cos\theta}{10+6cos\theta}d\theta = 0 \quad (22)$$

3.2 Example 2

The second example is to calculate the integral

$$I_1 = \oint \frac{e^{z-1}}{z-1}dz \quad (23)$$

(C:$|z|=2$), and then prove that

$$I_0 = \int_0^{\pi} \frac{e^{2cos\theta-1}\left[2cos(2sin\theta) - cos\theta cos(2sin\theta) + sin\theta sin(2sin\theta)\right]}{5-4cos\theta}d\theta = \frac{\pi}{2} \quad (24)$$

Proof. The author starts from finding this integral I_1

$$I_1 = \oint \frac{e^{z-1}}{z-1}dz \quad (25)$$

The integral is a fraction, and its denominator is $z-1$. When z equal to one, it can get that $z-1=0$,the denominator is equal to zero, and this fraction does not exist.

Next, it focuses on the contour of z , the contour is $|z|=2$, it contains the point $z=1$. It means that the fraction does not exists on its contour when $z=1$. As a result, $z=1$ is the only singular point of the integral $\oint \frac{e^{z-1}}{z-1}dz$ in the domain C. According to the theorem, the former integral $\oint \frac{e^{z-1}}{z-1}dz$ equal to $2\pi i$ because

$$I_1 = \oint \frac{e^{z-1}}{z-1} dz = 2\pi i e^{z-1} |z=1| = 2\pi i \quad (26)$$

Also, let $z = 2(cos\theta + isin\theta)$ $(-\pi \leqslant \theta \leqslant \pi)$, the integral $I_1 = \oint \frac{e^{z-1}}{z-1} dz$ can be rewrote in this form

$$I_1 = \int_{-\pi}^{\pi} \frac{e^{2cos\theta + 2isin\theta - 1}(-2sin\theta + 2icos\theta)}{2cos\theta + 2isin\theta - 1} d\theta \quad (27)$$

Then, it multiplies expression by $2cos\theta - 2isin\theta - 1$ and simplifies. As a result, the function can be divided to two integrals $I_1 = I_2 + I_3$, in which

$$I_2 = \int_{-\pi}^{\pi} \frac{e^{2cos\theta - 1}[2sin\{\theta cos(2sin\theta\}) + 2cos\theta sin(2sin\theta) - 4sin(2sin\theta)]}{5 - 4cos\theta} d\theta \quad (28)$$

and

$$I_3 = 2i \int_{-\pi}^{\pi} \frac{e^{2cos\theta - 1}[2cos(2sin\theta) - cos\theta cos(2sin\theta) + sinsin(2sin\theta)]}{5 - 4cos\theta} d\theta \quad (29)$$

The author now focuses on the second item I_3, and the range of second item I_3 is from $-\pi$ to π. In advance, simplifying the range of the integral, it can be rewritten as

$$I_4 = 4i \int_0^{\pi} \frac{e^{2cos\theta - 1}[2cos(2sin\theta) - cos\theta cos(2sin\theta) + sinsin(2sin\theta)]}{5 - 4cos\theta} d\theta \quad (30)$$

The value of this integral is equal to

$$I_4 = 4i \int_0^{\pi} \frac{e^{2cos\theta - 1}[2cos(2sin\theta) - cos\theta cos(2sin\theta) + sinsin(2sin\theta)]}{5 - 4cos\theta} d\theta = 2\pi i \quad (31)$$

So, The integral's value

$$I_0 = \int_0^{\pi} \frac{e^{2cos\theta - 1}[2cos(2sin\theta) - cos\theta cos(2sin\theta) + sinsin(2sin\theta)]}{5 - 4cos\theta} d\theta = \frac{\pi}{2}. \quad (32)$$

3.3 Example 3

In this example, the author shall apply the Cauchy-Gour sat theorem to calculate the integral of the below items. They have a contour C which is a circle of radius one, and the functions are

$$f(z) = \frac{1}{z^2 + 2z + 2} \quad (33)$$

and

$$f(z) = log(z+2). \quad (34)$$

Firstly, for Eq. (33), $f(z)$ is a kind of fraction, and it focus on the fraction's denominator $z^2 + 2z + 2$, it converts the form of its denominator, it equals to.

$$z^2 + 2z + 2 = (z+1)^2 + 1 \quad (35)$$

Also, the square of the $z+1$ must more than or equivalent to zero, which results that denominator of fraction must greater than or equal to one. And the contour is the circle $|z| = 1$, it means that the denominator doesn't be zero on this contour.

Accordingly, the function must exist on this contour. So, the function is analytic on its contour anywhere. The function satisfies the condition of Cauchy-Gour sat theorem. As a result:

$$\oint f(z) dz = \oint \frac{1}{(z+1)^2 + 1} dz = 0 \quad (36)$$

Secondly, for Eq. (34), this function belongs to a form of logarithm, and associated with the property of logarithm, $z + 2 \geqslant 0$. In the same way, the contour of the function is the circle $|z| = 1$. So, $z + 2 > 0$, the function doesn't have singular points. The function is analytic on its contour anywhere. The function satisfies the condition of Cauchy-Gour sat theorem. As a result:

$$\oint f(z) dz = \oint log(z+2) dz = 0 \quad (37)$$

4 Conclusion

The article explores the Cauchy integral formula, that is vital in calculus and complex functions. And concentrated on the Cauchy integral theorem and Cauchy Gour sat theorem, they are main theorems in the formula. Also, the article provides the verification of these theorems based on Cauchy-Riemann theorem and analytic functions. And the article discusses and elaborates the uses of the formula in the field of functions. A few examples shown in this article also demonstrate the application of the formula, such as relationship between the formula and the analytic functions. It not only simplifies the process of problem solving, but also furnishes a new way to analyze the problems. Despite the strict restrictions that the integral must be analytic or must exists the singular points to simplify the integral, by using this formula of Cauchy integral it can be generalize. The author also desires further research and survey on the theorem, such as the Cauchy residue theorem. It is based on the former, because in the theorem, it offers the concept of the singular points. As the further research of the singular points, the concept of the residue can be defined when z equal to the singular points z_0. And then, the Cauchy residue theorem can be gained.

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