Dean&Francis

ISSN 2959-6157

Control Variates in Option Pricing

Tongyue Huang[1+*],

Mingye Xiao[2+],

Ziyang Wang[3+]

[1]Guangzhou NO.3 High School, Guangzhou, 510120, China, huangtongyue070718@163.com,
[2]Hermann-Lietz Internat Schloss Bieberstein, Germany, 1057723126@qq.com
[3]Nanjing Bond International College, Nanjing, 210000, China, 3483867950@qq.com
+These authors contributed equally to this work and should be considered co-first authors.

Abstract:

—This paper explores the application of Monte Carlo simulation in option pricing, comparing its advantages and limitations to help better understanding the complexities of financial derivatives. Then the paper further investigates the variance reduction strategy known as control variates, which can check how big the error appears in Monte Carlo simulation and improves the accuracy and efficiency of Monte Carlo simulations. This study presents a comprehensive analysis of theoretical foundations of control variates and Monte Carlo simulation, and use Python examples demonstrating the practical applications in financial market. The results highlight the value of sophisticated computational methods for enhancing the accuracy of option pricing and, eventually, facilitating better informed decision-making in a financial environment which is more and more complex now.

Keywords:-Monte Carlo simulation; Control variates; Option pricing.

1. Introduction

In contemporary financial mathematics, the accurate pricing of options remains a pivotal challenge due to the inherent complexity of financial markets and the limitations of traditional analytical models. Options, integral to risk management and derivative trading strategies, necessitate precise valuation methods that can flexibly adapt to varying market conditions and instruments. The Monte Carlo simulation method emerges as a robust numerical technique capable of addressing these challenges by simulating numerous possible future scenarios, thereby providing a probabilistic framework for option pricing.

The Monte Carlo simulation relies on random sampling and computing power to estimate complex financial derivatives, such as call options, which depend on averaging their payoffs over simulated future states of the underlying asset. In this approach, one does not need the simplifying assumptions used in the analytical models since financial markets are intrinsically stochastic, and the further ahead one goes, the more uncertain the future price evolution of the assets is. By simulating many scenarios and aggregating the results, Monte Carlo methods offer a comprehensive assessment of option values, capturing the full range of potential outcomes and their associated probabilities.

This thesis proposal will delve into the use of Monte Carlo simulation for call option pricing to show how well it can be used to overcome some weaknesses of conventional models and increase pricing accuracy. The paper, therefore, reviews the practical implications of the Monte Carlo simulations in financial risk management through a careful review of methodologies and empirical studies on simulations. It evaluates certain variance reduction techniques, such as control variates, and demonstrates how improvements in simulation methodologies lead to more accurate and efficient estimation in option pricing.

2. Literature Review

Option pricing has long been one of the greatest challenges to be encountered in financial mathematics, since under an uncertain market environment, hedging and speculation predicate on it. Traditional models, such as the Black-Scholes model, prescribe the general framework but cannot capture the complexity of modern financial markets. In those models, volatility, risk-free interest rates, and returns of assets are assumed to be constant and normally distributed, respectively. These assumptions very seldom correspond to the real dynamics of markets.

In order to overcome these limitations, Monte Carlo simulation is increasingly used because it is a strong numerical technique that is able to handle stochastic character financial assets and market conditions. Monte Carlo simulation gives several random paths for the price of the underlying asset based on specified parameters, including volatility and drift, where option values are calculated by averages over the payoffs over those random paths. This approach allows the incorporation of volatility in the market, fluctuation in interest rates, and nonlinear payoffs in a more articulate manner. It provides a more realistic valuation of options.

Several studies have shown that the Monte Carlo simulation improves accuracy in option pricing over the traditional models currently used. It has also been highlighted, for example, by Haug and Taleb (2011), that there are a great number of advanced payoffs and non-normal distributions that are handled efficiently by Monte Carlo methods, given better estimates of pricing in volatile markets [1]. On the other side, Andersen and Broadie (2004) discuss a Monte Carlo simulation with several methods for creating controls of reduction techniques, that is, control variates and antithetic variates, which improve the computational efficiency and reduce the estimation variances for it [2].

Recent advances in computing power, along with algorithmic techniques, have further enhanced this applicability of Monte Carlo simulation in financial modelling. Being able to simulate a wide range of scenarios and incorporate real-time market data has made Monte Carlo methods indispensable for risk management and derivative pricing strategies [3].

Option pricing, in the last few decades, has been a dynamic field where improvements mainly come through computational methods and possible refinements of classic models. Among the developments that have been major trumpeters, the use of Monte Carlo simulations in financial modelling has been indispensable. The Monte Carlo methods pioneered by Boyle were followed to emerge in the limelight owing to great flexibility in handling complex derivatives featuring path dependency [4]. These techniques have been further refined into, among others, the works of Glasserman to introduce advanced methods of variance reduction in enhancing computational efficiency and accuracy in option pricing [3].

An important limitation of traditional models, emphasized in contemporary research, is the assumption of constant volatility. This weakness was surpassed by the stochastic volatility models; the most well-known is the Heston model, which models volatility as a stochastic process in itself [5]. Further work on multi-factor stochastic volatility models was conducted by Fouque, Papanicolaou, and Sircar for more realistic market behavior, especially in very volatile markets, enabling the better pricing of options and other derivatives [6].

Recent work by Broadie and Kaya focuses on enhancing the computational efficiency of Monte Carlo simulations under the Heston stochastic volatility framework [7]. Their study develops an exact simulation method for the Heston model, which drastically reduces the bias and variance typical with traditional Monte Carlo methods. By incorporating control variates, their approach has been shown to yield more accurate option prices while being computationally feasible.

The control variates technique, crucial for developing the variance reduction of Monte Carlo estimates, has first been applied to financial problems by Broadie and Glasserman and since then is extensively studied [8]. Very recently, Giles further developed the framework of variance reduction techniques through the multilevel Monte Carlo approach, which yet again increases the efficiency of Monte Carlo simulations via control variates between simulations at different levels [9].

Machine learning techniques have also begun to emerge as one of the promising directions for the improvement of methods of option pricing in financial modelling. For instance, Bayer, Horvath, and Muguruza investigated deep learning architectures in combination with traditional Monte Carlo and stochastic volatility models and showed that neural networks can be used for improving both speed and accuracy in option pricing models.

In sum, the literature brings out Monte Carlo simulation as both flexible and powerful in option pricing within modern financial mathematics. By embedding a probabilistic framework embracing the complexity of financial markets, Monte Carlo methods enjoy considerable advantages over more traditional analytical models, opening options

ISSN 2959-6157

for more valid and reliable option pricing strategies.

3. Methodology

The method adopted for this paper is the Monte Carlo simulation, considered one of the most powerful numerical techniques in financial mathematics in options pricing. Monte Carlo simulation may avail a strong probabilistic framework that fits very well with the stochastic characteristics following from financial assets and market variables; it therefore gives a more realistic estimation of option values than those from traditional analytical models.

Monte Carlo simulation generally generates a large number of random paths that the price of the underlying may take, usually based on processes such as the geometric Brownian motion. This approach enables one to approximate the value of option payoffs over an enormously wide range of simulated scenarios that grasp the whole gamut of possibilities. The core of this approach would be in computing the expected payoff in each scenario and combining these payoffs to obtain the present value of the option.

To enhance the efficiency of Monte Carlo simulations, control variates are employed as a variance reduction technique. Control variates are developed by the introduction of an auxiliary random variable correlated with the key variable of interest. In option pricing, by using control variates, the accuracy of the Monte Carlo estimates can be refined through the adjustment of simulated payoffs according to historic data and performance linked to financial assets.

First, one must validate the results from the Monte Carlo simulation. The sensitivity analysis will show how a variation in some of the input parameters regarding volatility, interest rates, and time to maturity can yield different option prices. Hence, this would portray the robustness of the results concerning various market scenarios and parameters.

By nature, Monte Carlo simulations are very computational; hence, efficient algorithmic implementation combined with optimization techniques allows for effective management of computation time and resources. For instance, parallel computing or GPU acceleration might be considered for large datasets or in the case of more complex derivative structures to accelerate the simulations.

In conclusion, by using advanced numerical methods and applying various techniques for variance reduction, such as a control variates technique, this study tries to contribute to further development of academic achievements in financial mathematics and risk management practices.

4. Verification

First, there is a general formula in Monte Carlo estimators.

$$q = \frac{1+r-d}{u-d} \tag{1}$$

(r means the interest rate; d means the stock downside multiplier; u means the stock up multiplier.)

The Monte-Carlo approach is that it generates many coin tosses (i.e., stock price) and then computes the empirical average. I.e., this work has examples f_1, ..., f_N of F (S_1) for some N. Then

$$X_0 \approx \frac{1}{1+r}\left(\frac{1}{N}\sum_{i=1}^{N} f_i\right) \tag{2}$$

With the Monte Carlo technique, this work can use it in option pricing. Suppose the pairs (X_i, Y_i), i=1, ..., n is i.i.d. (independent and identically distributed) sequence of random variables with the same distribution as X & Y. Here are some formulas about control variates. Suppose the expectation E[Y] is known. The control-variates estimator with parameter b of E[X] is defined by

$$\begin{aligned}\bar{X}_n(b) &= \bar{X}_n - b\left(\bar{Y}_n - E[Y]\right) = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - b\left(Y_i - E[Y]\right)\right)\\ &= \frac{1}{n}\sum_{i=1}^{n} X_i(b)\end{aligned} \tag{3}$$

Note that the observed error $\bar{Y}_n$-E[Y] is used to control the estimation of E[X]. The mean of the control-variates estimator is

$$E\left[\bar{X}_n(b)\right] = \frac{1}{n}\sum_{i=1}^{n} E\left[X_i(b)\right] = E[X] \tag{4}$$

The variance of the control-variates estimator is

$$\begin{aligned}Var\left(\bar{X}_n(b)\right) &= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i(b)\right) = \frac{1}{n} Var\left(X_i(b)\right)\\ &= \frac{1}{n}\left(Var(X) - 2bCov(X,Y) + b^2 Var(Y)\right)\end{aligned} \tag{5}$$

This variance is a function in b and this work want to minimize it with respect to b (recall: this work is allowed to choose b). By setting the derivative in b equal to zero this work can get the value b^* that minimizes the variance $Var(\bar{X}_n(b))$. This value is given by

$$b^* = \frac{Cov(X,Y)}{Var(Y)} \tag{6}$$

Substituting b^* for b, this work obtains

$$Var\left(\bar{X}_n(b^*)\right) = \frac{1}{n}\left(Var(X) - \frac{Cov(X,Y)^2}{Var(Y)}\right) \tag{7}$$

This expression and the fact that

$$Var\left(\bar{X}_n\right)=\frac{1}{n}Var\left(X\right) \tag{8}$$

imply that

$$\frac{Var\left(\bar{X}_n\left(b^*\right)\right)}{Var\left(\bar{X}_n\right)}=1-\frac{Cov\left(X,Y\right)^2}{Var\left(X\right)Var\left(Y\right)} \tag{9}$$

There is an example to show how control variates help reduce the variance of Monte Carlo estimators to improve the quality of the approximation. Suppose S0 - the true price - equals to 70, the strike price equals to 60, and the period equals to 5. Based on these data, the Monte Carlo estimate of the call price equals to 32.297. To observe the trend in the estimated option value, python is used in this work to generate some random numbers to draw a picture about the rolling average of approximated call value. As shown in Figure 1, the x-axis means number of samples, and the y-axis means approximated call value. Figure 1 shows that the mean values of both lines are similar, but the variance of blue line is much larger than the orange line. Specially, the blue line overestimates the call value when the sample size is not large enough, and as the sample size increases, the approximated call value is closer to the real value. By contrast, the orange line always closer to the real value than the blue line.

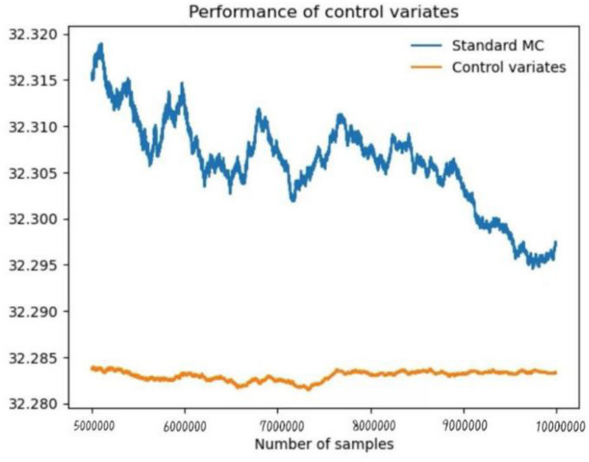

Figure 1. A comparison between standard Monte Carlo estimator and control variates.

Figure 2 shows how many reduction control variates can do with different strike price. Although the reduction becomes smaller when the strike price becomes larger, the reduction is still large enough to help this work approximate the real call value.

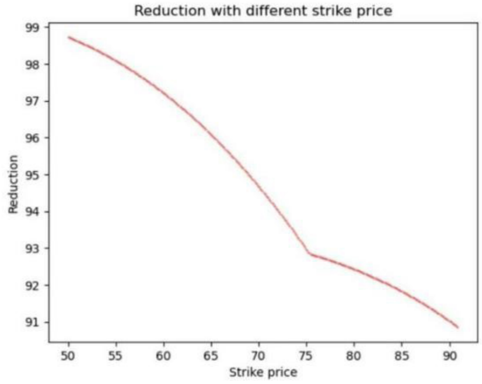

Figure 2. Reduction changes with different strike price.

5. Conclusion

The empirical study in this thesis proposal aims to apply Monte Carlo simulation to assess the efficacy of option pricing, specifically focusing on call options, in dynamic financial markets. The study will be structured to evaluate the performance of Monte Carlo methods in capturing the value of options under varying market conditions and parameter settings.

The Monte Carlo experiment will be built around generating a large number of random paths for the price of the underlying asset modelled under stochastic processes, such as geometric Brownian motion. Realistic market dynamics are captured by suitably calibrating parameters such as volatility, risk-free interest rates, and time to maturity using historical market data with the help of standard financial models.

The accuracy and reliability of the Monte Carlo estimates are investigated in this study for some metrics, considering the mean squared error, bias, and dispersion. Provided metrics will give an important insight into the precision of the option pricing estimate by Monte Carlo simulations in comparison with theoretical values and market observations.

The biggest part of the project involves comparing the Monte Carlo estimates using control variables with those that do not employ control variables. In these various simulations, control variates shall be applied to reduce the variance by making use of either historical data or even the performance of related assets. This provides a comparison that shows how efficient the methods for variance reduction are in enhancing the accuracy and efficiency of Monte Carlo simulations of option price.

The sensitivity analysis, considering how the alteration in input parameters—a change in volatility, interest rate, or prices of the underlying asset—changes the option prices, would therefore give a better insight into how robust the

ISSN 2959-6157

Monte Carlo simulations could be under different market scenarios and settings of parameters. This will therefore enhance the credibility and applicability of the study.

This work wants to give above all some meaningful development to the understanding of the efficiency of the Monte Carlo method and of its practical applicability in financial mathematics and in risk management by means of advanced numerical techniques and exhaustive sensitivity-variance analysis.

6. Acknowledgments

All authors, Tongyue Huang, Mingye Xiao, and Ziyang Wang, contributed equally to this work. They can be considered as co-first authors.

References

[1] Haug, E. G., & Taleb, N. N. (2011b). Option traders use (very) sophisticated heuristics, never the Black–Scholes–Merton formula. Journal of Economic Behavior & Organization, 77(2), 97–106. https://doi.org/10.1016/j.jebo.2010.09.013

[2] Andersen, L., & Broadie, M. (2004). Primal-Dual simulation algorithm for pricing multidimensional American options. Management Science, 50(9), 1222–1234. https://doi.org/10.1287/mnsc.1040.0258

[3] Glasserman, P. (2003). Monte Carlo methods in financial engineering. In Applications of mathematics. https://doi.org/10.1007/978-0-387-21617-1

[4] Boyle, P. P. (1977). Options: A Monte Carlo approach. Journal of Financial Economics, 4(3), 323–338. https://doi.org/10.1016/0304-405x(77)90005-8

[5] Heston, S. L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. Review of Financial Studies, 6(2), 327–343. https://doi.org/10.1093/rfs/6.2.327

[6] Fouque, J., Papanicolaou, G., & Sircar, K. R. (2000). Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press.

[7] Broadie, M., & Kaya, Ö. (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. Operations Research, 54(2), 217–231. https://doi.org/10.1287/opre.1050.0247

[8] Broadie, M., & Glasserman, P. (1996). Estimating security price derivatives using simulation. Management Science, 42(2), 269–285. https://doi.org/10.1287/mnsc.42.2.269

[9]Giles, M. B. (2008). Multilevel Monte Carlo Path Simulation. Operations Research, 56(3), 607–617. https://doi.org/10.1287/opre.1070.0496