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Applications of Harmonic Analysis in Quantum Computing

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Abstract:
The quantum Fourier transformation has been demonstrated to be a useful tool in dealing with quantum-mechanical problems. This paper discusses how quantum Fourier transformation is applied in finding the period of functions within Shor's algorithm, allowing the algorithm to be much more efficient in factorizing large integers than classic algorithms. This paper also discusses how Grover's algorithm uses the concept of harmonic process, where amplitude amplification is likened to harmonic oscillation for improving the efficiency of searching unsorted data base. The paper further explores the application of harmonic analysis in quantum error correction, particularly in stabilizer codes. The paper also introduces wavelet analysis as an alternative approach for detecting and correcting localized quantum errors. Finally, the paper extends the discussion into the potential future directions of harmonic analysis in quantum computing, such as extending Fourier transformation to non-abelian groups and using spherical harmonics in quantum algorithms which may optimize current quantum algorithms and solve currently unsolved quantum questions.

Keywords: Fourier Analysis, Harmonic Analysis, Quantum Computing.

1. Introduction

Quantum computing is an increasingly new area of study that has been developed less than 50 years. There is a rapid increase in interest in this field because quantum computing had shown its ability to solve complexed problems much faster than classical computers. The power of quantum algorithms lies in their ability to use the principles of quantum mechanics, such as superposition and entanglement, allowing them to perform computations in parallel and solve problems with exponential speedup. A crucial component of these quantum algorithm is harmonic analysis, particularly the use of Fourier transform, and wavelet transform.

Harmonic analysis a field of study in mathematics that studies representing signals and functions as combinations of basic waveforms, which plays a central role in quantum algorithms. This paper investigates the use of harmonic analysis in several key quantum algorithms, including Shor's algorithm for large integer factorization and Grover's algorithm for unsorted database search which uses harmonic analysis to achieve significant computational advantages over other classical algorithms. In addition to the application in quantum algorithms, harmonic analysis is also applied in quantum error correction. Quantum error correction is one of the major challenges in quantum computing and a significant part for any quantum technologies. This paper explores how harmonic analysis and wavelet techniques can be employed in stabilizer codes and error correction protocols to identify and correct errors in quantum systems.

By exploring these areas, the paper seeks to provide a comprehensive overview of how harmonic is used in quantum computing, surpassing other classical algorithms. The paper also highlights future directions in the field, such as extending harmonic analysis to non-abelian groups and employing spherical harmonics for improving efficiency in quantum chemistry and material science.

2. Background Information

2.1 Harmonic Analysis and Fourier Analysis

Harmonic analysis is a branch of mathematics that studies representing functions and signals as the superposition of simple waves. This is an extend of Fourier series and Fourier transforms, providing tools for analysing functions and operators. Harmonic analysis is very important in many fields including signal processing, quantum mechanics and number theory [1].

One of the core concepts of harmonic analysis is Fourier series and Fourier Transform. Fourier series states that for any periodic function $f(x)$ with period 2π can be represented as an infinite sum of sines and cosines

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n cos(nx) + b_n \sin(nx)), \quad (1)$$

where the coefficients a_n and b_n are defined as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) cos(nx)dx, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \quad (2)$$

Fourier transform states that for non-periodic functions $f(x) \in L^1(\mathbb{R})$, the Fourier transform is defined as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx. \quad (3)$$

The inverse Fourier transform is defined as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega. \quad (4)$$

Harmonic analysis often operates within Hilbert spaces. For functions $f, g \in L^2(\mathbb{R})$, define the inner product as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g^*(x) dx, \quad (5)$$

where $g^*(x)$ denotes the complex conjugate of $g(x)$. There are some very important properties of Harmonic Analysis. The first one is linearity, that the Fourier transform is a linear operator:

$$F\{af(x) + bg(x)\} = a\hat{f}(\omega) + b\hat{g}(\omega). \quad (6)$$

The second one is the convolution theorem, that the Fourier transform of a convolution is the pointwise product of Fourier transform:

$$F\{f * g\} = \hat{f}(\omega)\hat{g}(\omega), \quad (7)$$

where $(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt.$ The third property is Parseval's identity, that the total energy of a signal is preserved under the Fourier transform:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega. \quad (8)$$

2.2 Wavelet Analysis

Another very important aspect of harmonic analysis is wavelet analysis. Wavelet analysis allows to decompose a function into components that are both localized in time and frequency. Unlike the Fourier transform, which represents signals as sum of sine and cosine waves with global frequency components, wavelet analysis uses functions called wavelets that are localized in both time and scale, making it particularly useful for signals that have transient or non-stationary characteristics. Wavelet analysis involves wavelet transform, which mainly involves 2 transformations. The first one is continuous wavelet transform (CWT) of a function $f(t)$ is defined by

$$W(a,b) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) dt \quad (9)$$

where $f(t)$ is the input signal, $\psi(t)$ is the mother wavelet, a is the scale parameter which controls the frequency and b is the translation parameter, which controls the time shift. Another transformation is discrete wavelet transformation (DWT), which is defined by

$$W_{j,k} = \int_{-\infty}^{\infty} f(t) \psi_{j,k}(t) dt, \quad (10)$$

where j and k are integers indices that represent scale and translation and $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ is the scaled and translated wavelet function. The DWT produces a hierarchical representation of the signal, which allows a signal to be represented at different levels of detail using scaling (low-pass) and wavelet (high-pass) coefficients by using multi-resolution analysis [1].

3. Algorithms and Quantum Error

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Correction

3.1 Quantum Fourier Transform in Shor's Algorithm

Shor's algorithm is an algorithm that aims to solve problems of large integer factorization. Shor's algorithm has many implications in cryptography, particularly for cryptosystems that rely on the hardness of factoring large numbers, which include the most widely used RSA system.

An important process in Shor's Algorithm is to find the period of the function $f(x) = a^x modN$ where the period is the smallest positive integer r such that $a^r \equiv 1 modN$. Quantum Fourier Transform is the core of finding such r. In order to find the period, the first step is to prepare a superposition of all possible input $|x?$, which can be written as $\frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x?$, where Q is a power of 2 such that $Q > N^2$. After this, Shor's algorithm evaluates f for all x and stores the result in the second register. This changes the quantum states to

$$\frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x>|f(x)>. \quad (11)$$

Since f is periodic with the period r, one can rewrite the state into

$$\frac{1}{\sqrt{Q}}\sum_{y=0}^{r-1}\sum_{k=0}^{\left|\frac{(Q-y-1)}{r}\right|}|y+kr\rangle|f(y)\rangle. \quad (12)$$

When measuring the second register, the superposition collapses to a state where all x correspond to the observed $f(y)$. Thus, the state becomes

$$\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}|y+kr\rangle, \quad (13)$$

where $M = \left\lfloor \frac{(Q-y-1)}{r} \right\rfloor$. Shor's algorithm now applies the Quantum Fourier Transformation. The Quantum Fourier Transformation (QFT) is defined as

$$QFT(|x\rangle) = \frac{1}{\sqrt{Q}}\sum_{z=0}^{Q-1}e^{\frac{2\pi ixz}{Q}}|z. \quad (14)$$

Applying Quantum Fourier Transformation to $\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}|y+kr\rangle$, the state transforms to

$$\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}\left(\frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}e^{\frac{2\pi i(y+kr)z}{Q}}|z?\right), \quad (15)$$

which can be simplified into

$$\frac{1}{\sqrt{QM}}\sum_{z=0}^{Q-1}e^{\frac{2\pi iyz}{Q}}\left(\sum_{k=0}^{M-1}e^{\frac{2\pi ikrz}{Q}}\right)|z\rangle. \quad (16)$$

Notice $\sum_{k=0}^{M-1}e^{\frac{2\pi ikrz}{Q}}$ is a geometric series, so

$$\sum_{k=0}^{M-1}e^{\frac{2\pi ikrz}{Q}} = \frac{1-e^{\frac{2\pi irMz}{Q}}}{1-e^{\frac{2\pi irz}{Q}}}. \quad (17)$$

The magnitude of this expression is optimized when $e^{\frac{2\pi irz}{Q}} \approx 1 \cdot e^{\frac{2\pi irz}{Q}} = 1$ when $z = mQ/r$. This tells people that the value of z leading to constructive inference are multiples of Q/r. After z is measured, it can be seen that $z \approx mQ/r$ for some integer m, meaning that $z/Q \approx m/r$, thus deducing the period r [2].

In some cases, the function has a sparse frequency spectrum, meaning that the Fourier coefficients of the function are mostly zero. This kind of function is common in period-finding problems. For these functions, instead of calculating the entire Fourier transform, which would take a time complexity of $O(NlogN)$ operations in the classical case and $O(n^2)$ in the quantum case where $N = 2^n$, Sparse Fourier Transform reduces the time complexity into $O(klogN)$ where k is the sparsity (the number of significant Fourier coefficient). Sparse Fourier Transform includes the randomized subsampling and aliasing of the original signal $x[n]$ of length N. Mathematically it is $x_S[n] = \{x[n] : n \in S\}$, for $k = 0,1,\ldots,N-1$ and S is a subset such that $S \subset \{0,1,\ldots,N-1\}$ with $|S| = m$ where $m \ll N$. After this, Sparse Fourier Transform algorithm would apply chirp filters or Gaussian filters to the subsampled data. These filters are used to supress noise. Chirp function is of the form $Chirp(n) = e^{i\alpha n^2}$ and Gaussian filter is of the form $G(n) = e^{\frac{-n^2}{2\sigma^2}}$. Finally, given the filtered and subsampled signal $x_S[n]$, one can compute the non-zero or significant Fourier coefficient by [2]

$$\hat{X}[k] = \sum_{n\in S}x[n]e^{\frac{-2\pi ikn}{N}}. \quad (18)$$

3.2 Harmonic Process in Grover's Algorithms

Grover's algorithm is an algorithm that provides a faster solution to unstructured searching problems than classical algorithms. In classical algorithm, the time complexity of finding a marked element in an unsorted database of N elements is $O(N)$. While the time complexity of Grover's algorithm is $O\left(\sqrt{N}\right)$. The main idea of Grover's algorithm is to amplify the amplitude of the marked state by a series of quantum operations.

Mathematically, the amplitude amplification in Grover's algorithm can be interpret by the rotations in the Hilbert space. Let's assume that there is an unsorted database with N elements, and one wants to search a particular element $|w?$. In Grover's algorithm, the algorithm first initialize an equal superposition of all possible state $|s?$

$$|s? = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle. \quad (19)$$

Grover's algorithm then applies an oracle operation O_w, which marks $|w\rangle$ by

$$O_w|x\rangle = \begin{cases} -|x\rangle if x = w \\ |x\rangle if x \neq w \end{cases}. \quad (20)$$

This operation can be understood as a reflection across the $|x \neq w?$ states in the Hilbert space. After applying the oracle, Grover's algorithm uses Grover iteration to amplify the probability amplitude of the marked state. The state $|s?$ can be broken down into the component in the direction to $|w?$ and the component in the direction orthogonal to $|w?$, denoting it as $|r?$. $|r?$ represents all other states. One can express $|s?$ as $|s? = \alpha|w? + \beta|r?$ where α is the amplitude of the target state and β is the amplitude of all other states. The Grover iteration consists of 2 operations: O_w and diffusion operator D. The combination of O_w and D perform a rotation in the plane spanned by $|w?$ and $|r?$ with the angle $\theta = 2arc\sin\left(\frac{1}{\sqrt{N}}\right)$. The amplitude amplification process is similar to a harmonic oscillation mathematically. After each Grover iteration, the amplitude of $|w?$ increases while the amplitude of other states decreases. The probability of measuring the $|w?$ after k iterations is

$\sin^2\left(\left(k+\frac{1}{2}\right)\theta\right)$. The probability of $|w?$ would increase as the algorithm applies Grover iterations until the angle reaches $\frac{\pi}{2}$. If too many iterations are performed such that the angle surpass $\frac{\pi}{2}$, the probability of $|w?$ would decrease. Grover's algorithm essentially uses the concept of repeated reflections and amplifications, which is a type of harmonic process [3].

3.3 Harmonic Analysis in Quantum Error Correction

A quantum error correction algorithm is an algorithm designed to detect and correct errors in quantum states caused by the noise in quantum system. Although Quantum Fourier Transform is not part of most standard error such as the Shor code and Steane code, it is especially useful in stabilizer codes. In stabilizer codes, the Quantum Fourier Transformation is used in the phase estimation algorithm to find the error syndrome [4].

The condition for applying stabilizer error correction is that different errors must produce distinguishable syndromes and that errors within the same equivalence class do not affect the logical state. Mathematically, the condition of applying stabilizer error correction can be written as

$$\langle\psi_i|E_k^\dagger E_l|\psi_j\rangle = \delta_{ij}C_{kl}, \quad (21)$$

where $|\psi_i\rangle$ and $|\psi_j\rangle$ are logical code states, E_l and E_k are error operators from the set of possible errors that the code is to correct, δ_{ij} is the Kronecker delta, ensuring that the inner product between 2 different logical states is zero, and C_{kl} is a constant depending on E_l and E_k.

In stabilizer code, given a state $|\psi\rangle \in C$ where C is the common eigenspace of a set commuting Pauli operator (stabilizers) and an error E where E is a Pauli operator, one can measure the stabilizers which would return either 1 or -1. If it is 1, then it indicates that the qubit is not affected by the error, whereas -1 indicates that the qubit has been affected by the error. If the error involves phase shifts, Quantum Fourier Transform is used in Quantum Phase Estimation which can determine the amount of phase shifted [5].

Besides the application of Fourier analysis in quantum error correction, some of the newest studies shows that wavelet analysis can be used in quantum error correction. Different from QFT, Wavelets analysis can be used for localized time-frequency analysis, which is better in detecting and correcting errors that are localized both in time and frequency domains. In classical signal processing, wavelet transforms would decompose a signal at various scales, providing information for both time and frequency. Similarly, in quantum computing, wavelet transform can be used for analysing quantum states at different scales. This can detect error that are not apparent in standard basis [6].

Mathematically, a wavelet transforms of a quantum state

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$|\psi\rangle$ can be written as

$$\left|\bar{\psi}\right\rangle = W|\psi\rangle, \tag{22}$$

where W is the wavelet transform operator. The operator W is constructed using a set of orthonormal wavelet basis $\{\varphi_j\}$ and scaling functions $\{\psi_j\}$ such that

$$\int \varphi_j(x)\varphi_k^*(x)dx = \delta_{jk} \tag{23}$$

and

$$\int \psi_j(x)\psi_k^*(x)dx = \delta_{jk}. \tag{24}$$

In order to implement wavelet transforms in a quantum circuit, the wavelet transform need to be decomposed into a sequence of quantum gates, like what is done for QFT. For example, the discrete wavelet transform can be implemented using the combination of Hadamard gates and controlled operations.

When an error E acts on a quantum state $|\psi\rangle$, the state becomes $E|\psi\rangle$. By applying the wavelet transform, the state becomes

$$\left|\psi_E\right\rangle = WE|\psi\rangle = WEW^{\dagger}W|\psi\rangle = E\left|\psi\right\rangle, \tag{25}$$

where $E = WEW^?$ is the error operator in the wavelet-transformed basis. The transformed error operator E can detect and correct localized error. For example, suppose E is a bit-flip error on qubit i, represented by the Pauli X operator X_i. In the wavelet basis, the error E would affect specific wavelet coefficient that corresponds to qubit $i's$ scale, allowing the error to be detected and corrected.

The condition for successful error correction using wavelet is similar to that of stabilizer codes but using wavelet basis, that is

$$\widetilde{\psi_i}\left|\widetilde{E_k^{\dagger}}\widetilde{E_l}\right|\widetilde{\psi_j} = \delta_{ij}\widetilde{C_{kl}}, \tag{26}$$

where $|\widetilde{\psi_i}$ and $|\widetilde{\psi_j}$ are logical code states in the wavelet basis, E_l and E_k are transformed error operators and C_{kl} is a constant depending on E_l and E_k [7].

3.4 Future Directions

Traditional quantum algorithms like Shor's algorithm applied the QFT over abelian groups (for example $\mathbb{Z}_n$). Extending the abelian group into non-abelian groups could lead to new algorithms, optimizing existing ones. Notice there is already existing Fourier transform over non-abelian groups. For a finite group G, the Fourier transform of a function $f: G \to C$ is given by

$$\hat{f}(\rho) = \sum\nolimits_{g\in G} f(g)\rho(g). \tag{27}$$

Algorithms that are applied over non-abelian groups can solve problems like hidden subgroup problem for non-abelian groups, which has still remained unsolved [8]. Applying spherical harmonics could optimize algorithms involving rotational symmetries like problems in quantum chemistry and material science. Spherical harmonics are eigenfunctions of the Laplace operator on the sphere and form an orthonormal basis for square-integrable functions on the sphere. Quantum states representing functions on spheres can be easier to analyse by applying spherical harmonics

$$|\psi = \sum_{1=0}^{\infty}\sum_{m=-1}^{1} c_{1m}|Y_{1m}, \tag{28}$$

where c_{lm} are coefficients and $|Y_{1m}$ is the spherical harmonics.

4. Conclusion

In conclusion, this paper explores the application of harmonic analysis in quantum computing, outlining its role in various quantum algorithms and quantum error correction. Harmonic analysis is a very important field of study in representing and analyzing quantum states. Especially the quantum Fourier transformation, is the core of Shor's algorithm for large integer factorization and Grover's algorithm for searching a specific element in an unsorted data base. The harmonic analysis plays an important row in quantum computing by improving the efficiency of quantum algorithms and contributing to error correction methods. The application of the quantum Fourier transformation in algorithms such as Shor's and Grover's algorithm demonstrate its ability in solving complex computational problems that would be a challenge for classical systems. Additionally, the stabilizer codes and possible application of wavelet transform in quantum error correction codes shows the capability of harmonic analysis in solving quantum error corrections. The future directions of harmonic analysis in quantum computing includes extending Fourier analysis to non-abelian groups and applying spherical harmonics to solve more advanced quantum problems and optimizing existing algorithms in fields like quantum chemistry and material science. These potential directions highlight the importance of harmonic analysis in the future of quantum computing.

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