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Exploring Hahn-Banach Theorem: Insights into Functional Analysis and Topology

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Abstract:

This paper delves into the Hahn-Banach Theorem, elucidating its significance and applications in functional analysis and topology. It commences by highlighting foundational concepts in normed linear spaces and gradually builds towards a comprehensive exploration of the theorem's multiple forms. By reconstructing the proof from basic principles, the paper demonstrates the theorem's versatility in extending linear functionals from subspaces to universal sets, initially within real linear spaces and subsequently in complex scenarios. The proof employs techniques like sublinearity, seminorms, and Zorn's Lemma to articulate the theorem's efficacy in both real and complex linear spaces. This study also discusses the adaptation of the theorem in normed linear spaces, where the absence of an explicit dominating function presents unique challenges, and continuity replaces boundedness to establish the theorem's claims. Through rigorous analysis, the paper confirms the theorem's pivotal role in linking points and functions within normed spaces, thereby enhancing understanding of dual spaces and their topological implications. Additionally, the practical applications of the theorem in establishing the existence of linear functionals and exploring the relationship between points and functional norms in normed spaces are also detailed, underscoring the theorem's enduring relevance in mathematical discourse.

Keywords: Hahn-Banach theorem; normed linear space; functional analysis; topology.

1. Introduction

Linear algebra, a cornerstone of mathematical education, introduces students to fundamental concepts such as linear spaces and linear maps. Traditionally, these studies focus on finite-dimensional spaces, where the existence of a finite basis simplifies operations and theoretical explorations. However, as one advances into the realm of functional analysis, the infinite-dimensional spaces defy some of these simplifying features, such as the lack of a finite basis, which profoundly impacts the approach to studying

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linear structures. The theoretical frameworks developed for finite dimensions, involving methods like mathematical induction and proof by contradiction, often require reevaluation or adaptation in this broader context [1].
The evolution of these concepts into the analytical study of functions in real variables introduces additional complexity. For example, notions like the limit of a sequence and function continuity, which hold under the structure of , find new expressions and implications within the framework of topological spaces. Particularly, a normed linear space offers a rich structure for extending many of these theories due to its inherent topological properties [2, 3].
The early 20th century marked the beginning of a shift in how mathematical theories were applied, moving from traditional linear equations and integrals to more complex structures like normed linear spaces. A significant milestone was achieved in 1936 when Murray extended the Hahn-Banach theorem to complex linear spaces, broadening its applicability across various mathematical disciplines [4].
This paper is divided into two main sections that explore the structural dynamics and theoretical applications of the Hahn-Banach theorem within normed linear spaces. The first section critically reviews structural results in normed linear spaces, setting the stage for a deeper understanding of the theorem's implications. The second part delves into the classical forms of the Hahn-Banach theorems, discussing their foundational principles and their practical applications within these spaces. Through this detailed examination, the paper aims to provide a comprehensive overview of the theorem's versatility and its enduring relevance in advancing the field of mathematical analysis.

2. Preliminary Concepts

This part reviews (without proof) the basic notions in normed linear space, aiming to lay a foundation for Hahn-Banach Theorem later in this paper.
In the definition above $(V, \|\bullet\|)$ is called a normed linear space, which can induce a metric by the norm of the difference of two vectors. Furthermore, it can induce a topology through this metric. Therefore, all notions in topology can be applied to normed linear space, such as openness, closeness, compactness, continuity, convergence, etc.
Continuity is a commonly investigated concept in mathematical analysis, its definition in topology is as below.
The definition itself has some application in the Open Mapping theorem [5]. For example, determine the continuity of an inverse function.
In normed linear space, there are some alternative ways to judge continuity from other perspectives. Among them a significant proposition is that continuity is equivalent to boundedness, which allows one to transfer question involving continuity to finding an upper bound.
The topological dual space requires one more condition that such linear functionals need to be continuous. Note that dual space refers to topological dual space in the following text.
There are many studies on the relation between dual space and original space and building an isomorphism from original space to double dual space [6].

3. Hahn-Banach Theorem

First review some terminology involved in several forms of Hahn-Banach Theorem.
Definition 2.1(sublinear & seminorm): Moreover, if p also satisfies $p(\lambda x) = |\lambda| p(x)$ for any $x \in X$, $\lambda \in \mathbb{F}$, then p is called a seminorm.
To prove Hahn-Banach Theorem, one can use the following Zorn's Lemma.
Proof: see [7].
Several forms of Hahn-Banach Theorem are given below as theorem 2.3, 2.4, 2.5.
Proof: The statement is trivial if $Y = X$. Suppose Y is a proper subset of X. One can first linearly extend l to a subspace which contains Y while maintaining the property that l is dominated by p.

$$l(y + az) = l(y) + al(z) \quad (1)$$

Here $l(z)$ is an undetermined real number now. One can readily confirm that l is a linear functional on Z. Next to show $l(z)$ can be chosen to satisfy that l is dominated by p on Z.

$$l(y) + al(z) \leqslant p(y + az),\ \forall y \in Y,\ a \in \mathbb{R} \quad (2)$$

Since p is a sublinear, (2.2) can be reduced to the case when $a = 1$ and $a = -1$:
$l(y) + l(z) \leqslant p(y + z)$, $l(y') - l(z) \leqslant p(y' - z)$, $\forall y, y' \in Y$, this is equivalent to.

$$l(y') - p(y' - z) \leqslant l(z) \leqslant p(y + z) - l(y),\ \forall y, y' \in Y \quad (3)$$

Notice that y and y' are two different and independent variables, the leftmost term of (2.3) is a function of y' and rightmost term is a function of y.
In fact, for any $y, y' \in Y$, observing that z term is vanished when sum up the terms in the brackets of p.
Making a small deformation of the above inequality gives (2.4), and thus (2.3) is verified.
To see (2.2), it can be divided into three cases:
$a = 0$, trivial $a > 0$.

$$l(y)+al(z)=a(\frac{1}{a}l(y)+l(z))=a(l(\frac{1}{a}y)+l(z))\leqslant ap(\frac{1}{a}y+z)=p(y+az) \quad (4)$$

$a<0$ The above argument gives a particular extension of l. This theorem requires one to give a universal extension (i.e. extend l to X). To do this, one interesting idea is to consider the set consisting of all such extensions. The question is transferred to show the universal extension is one element in this set.

Define an order on S by $(Z_1,l_1)\leqslant(Z_2,l_2)$ if $Z_2\supset Z_1$ and $l_2|_{Z_1}=l_1$. For any chain $\{(Z_\alpha,l_\alpha)\,|\,\alpha\in I\}$ in S, let $Z=\bigcup_{\alpha\in I}Z_\alpha$. Define l' on Z by l' equals to l_α on Z_α. The linearity of l' is obvious and l' is dominated by p on Z. So $(Z,l')\in S$ and (Z,l') is an upper bound for chain $\{(Z_\alpha,l_\alpha)\,|\,\alpha\in I\}$ by construction. By Zorn's Lemma, S has a maximal element (Z_0,L). By previous argument, Z_0 must be the whole space X. Otherwise, l can be extend to a larger subspace than Z_0, which yields a larger element in S. (X,L) is the desired extension.

Remark: This proof draws inspiration from the ideas discussed on pages 19-21 in reference [8]. The primary utility of this theorem lies in its ability to extend a linear functional from a subspace to a universal space, showcasing an elegant result of existence under specific conditions. This reflects the mathematical concept of expanding local properties to a universal context. Theorem 2.3 applies specifically to linear spaces over the field (R). A pertinent question arises: does the theorem hold for the field (C)? The answer is affirmative, though it imposes a stricter requirement that the functional must be a seminorm.

To substantiate this, it suffices to demonstrate the scenario when the field is (C). This adaptation of the theorem involves extending the functional to the universal set, paralleling the form presented in Theorem 2.3. However, since Theorem 2.3 is framed within the context of the real numbers, an approach to address the complex case involves initially separating the real and imaginary parts of the complex numbers involved. This method allows the theorem's application to be adapted suitably for complex fields by building on the logical structure established for the real case..

Let $f=g+ih$, where g and h are both real-valued linear functions on Y. Next to find a relation between g and h so that f can be expressed by only one function. Substitute iy into the expression to get

$$g(iy)+ih(iy)=f(iy)=if(y)=i(g(y)+ih(y))=-h(y)+ig(y),\ \forall y\in Y \quad (5)$$

Taking real part on both sides can get $-h(y)=g(iy)$. So $f(y)=g(y)-ig(iy)$.

The plan is to use the previous result for g, the only remaining thing is to show that g is dominated by p. Actually $g(y)\leqslant|g(y)|\leqslant|f(y)|\leqslant p(y)$, $\forall y\in Y$. By theorem 2.3, there exists a $\mathbb{R}-$linear functional $G:X\to\mathbb{R}$ such that $G|_Y=g$ and G is dominated by p on X.

Next to construct the function F, since F needs to be the extension of f to the whole space, F should have a very similar structure with f. Notice that one has already got G as the extension of g just now.

Finally show that $|F|$ is dominated by p on X: This can be done by considering the exponential representation of $F(x)$ to remove the module. For any $x\in X$, let $F(x)=re^{i\theta}$. Since $|F(x)|$ is real and p is a seminorm on X,

$$|F(x)|=r=e^{-i\theta}F(x)=F(e^{-i\theta}x)=G(e^{-i\theta}x)\leqslant p(e^{-i\theta}x)=|e^{-i\theta}|\,p(x)=p(x) \quad (6)$$

Remark: This proof refers to the idea of page 144. It uses the method of separation of real and imaginary part and exponential representation in complex analysis [9].

When a linear space equips with a norm, it gets some extra properties. This theorem has another form on normed linear space as following, it does not require an explicit dominating function anymore, but it needs the local functional f to be continuous. Furthermore, the extended function keeps the norm unchanged.

Proof: This theorem also requires generalizing a local function f to the whole space X just like previous two results. The idea is again to use the previous theorem 2.4. However, there is no dominating function given in the conditions. To solve this problem, one can construct a dominated seminorm for f as following [10].

Can deduce that $\|f\|=\sup_{y\in Y,\|y\|=1}|f(y)|\leqslant\|F\|$. Combining these two results can get the desired equality.

Remark: The idea of constructing the dominated seminorm p in this proof refers to [9] page 145.

Hahn-Banach Theorem is frequently applied in normed linear space to find relations between points and functions based on theorem 2.5. Here are two corollaries.

Corollary 2.6: Let X be a normed linear space over $\mathbb{F}$

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and $x_0 \in X$. Then there exists $f \in X^*$ such that $\|f\|_{X^*} = 1$ and $f(x_0) = \|x_0\|$.

Proof: If $x_0 = 0$, the statement is trivial since one can take f to be an arbitrary unit vector in X^*. Suppose $x_0 \neq 0$. To use theorem 2.5, one needs to construct a subspace and a functional with norm 1 on it. A feasible way is to consider the linear span of the point x_0.

Let $Z = \{\lambda x_0 \mid \lambda \in \mathbb{F}\}$. Define $h: Z \to \mathbb{F}$ by $h(\lambda x_0) = \lambda \|x_0\|$ for any $\lambda \in \mathbb{F}$. One can readily confirm that h is linear. The boundedness of h is also necessary. In fact, for any $z \in Z$, $z = \lambda x_0$ for some $\lambda \in \mathbb{F}$. $|h(z)| = |h(\lambda x_0)| = |\lambda \|x_0\|| = |\lambda| \|x_0\| = \|\lambda x_0\| = \|z\|$. From this one can get $h \in Z^*$ and $\|h\| = 1$.

Remark: This corollary has a more concise form compared to original Hahn-Banach Theorem. It only requires a normed linear space and a point in it to get an existence result, which means that it is convenient to use this result in any situation involving a norm. Below is a direct corollary.

Proof: One direction is trivial, just by norm inequality, $\sup_{f \in X^*, \|f\|=1} |f(x_0)| \leq \|x_0\|$. For the other direction, one can find a particular f and apply the property of supremum. Combine these two inequalities can get desired result.

Remark: This corollary reveals the relation between the norm of a point and the continuous linear functional acting on this point. What makes it interesting is that the definition of norm of continuous linear functional is $\|f\| = \sup_{x \in X, \|x\|=1} |f(x)|$. Compare this with the formula in corollary 2.7 one can find that they look like just changing the position of point and functional.

Hahn-Banach Theorem has a generalized form in topological vector space.

4. Conclusion

This paper has comprehensively explored the Hahn-Banach theorem, an essential principle in functional analysis and topology, demonstrating its fundamental role and extensive applications within the framework of normed linear spaces. Through meticulous analysis, the paper not only revisits the foundational proofs of the theorem but also extends these concepts to complex and normed linear spaces. By delving into the practical aspects of the theorem, such as the extension of linear functionals and the relationship between point norms and functional norms, this study reinforces the theorem's profound impact on modern mathematical analysis. The discussions herein have elucidated the theorem's ability to generalize local properties to universal contexts, thereby providing crucial insights into the structure of dual spaces and the interplay between points and functionals. Looking forward, the paper sets the stage for further research in several promising directions. One such area involves exploring the Hahn-Banach theorem within the broader scope of topological vector spaces, where the interactions between topology and linear functional could yield new theoretical insights and applications. Additionally, the potential to apply these concepts to emerging fields such as data science, quantum computing, and complex systems theory presents an exciting frontier. Future studies could also focus on developing computational algorithms that leverage the theorem's principles to solve real-world problems in engineering and physics more efficiently. By expanding the applications of the Hahn-Banach theorem and exploring its implications in various modern contexts, ongoing research can continue to build on the robust foundation provided by this pivotal mathematical theorem.

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