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The Taylor Series and Its Application

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Abstract:

Taylor series, a fundamental concept in mathematics, expresses a function as an infinite sum of terms which is calculated from the values of its derivatives at a single point. Its applications span various fields, notably physics and finance, where it simplifies complex functions for analysis and problem-solving. This study employs a qualitative approach, reviewing existing literature on Taylor series applications. It analyzes case studies in physics, such as approximating motion equations, and in finance, like option pricing models. The method includes mathematical derivations and simulations to demonstrate efficacy. Findings indicate that Taylor series significantly enhance computational efficiency and accuracy. In physics, it aids in predicting particle behavior under varying forces. In finance, it provides valuable insights into the pricing of derivatives and risk assessment. The Taylor series proves to be an indispensable tool in both physics and finance, facilitating complex calculations and improving analytical capabilities. Its versatility underscores the importance of mathematical foundations in real-world applications.

Keywords: Taylor series; application; economy; approximation

1. Introduction

Taylor's theorem is a basic result in mathematics that allows functions to be approximated by polynomials. More precisely, it states that any sufficiently smooth function can be represented in the Taylor series around a point. This series is an infinite sum of terms computed from the function's value of the derivative at that point. Taylor's formula is of great use in the use of filters [1]. Taylor's formula can also be used in physics such as thermal energy, electromagnetic energy [2]. Taylor's formula can also be used for modeling optimization [3]. In classical mechanics, the Taylor series can be used to approximate the position, velocity, and acceleration of objects [4]. For instance, when analyzing the motion of a projectile, the motion's equation can be expanded using Taylor series to simplify calculations near a specific point, such as the initial position. The Taylor series can also help approximate electric and magnetic fields in complex configurations [5]. For example, when calculating the potential due to a point charge, the potential can be expanded around a reference point to simplify the analysis of electric fields in the vicinity of the charge.

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2. Statement of the Theorem

2.1 Basic theory

2.1.1 Fundamental Form

L Let h be a function that is n-times differentiable at a point a. Then, the Taylor series of h around a is given by:

$$h(a) + h'(a)(x-a) + \frac{h''(a)}{2!}(x-a)^2 + \frac{h'''(a)}{3!}(x-a)^3 + \ldots + \frac{h^{(n)}(a)}{n!}(x-a)^n + R_n(x) \quad (1)$$

where $R_n(x)$ is the remainder term, which can be expressed in several forms, including the Lagrange form:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, -\infty < x < \infty \quad (2)$$

for some c between a and x [6].

2.1.2 Proof of Taylor's Theorem

The proof of Taylor's Theorem can be derived using the Mean Value Theorem and induction. Here is a brief outline:

Base case (n = 1): For a function h that is differentiable, we can express h(x) as:

$$h(x)=h(a)+h'(a)(x-a)+R_1(x) \quad (3)$$

where $R_1(x)$ can be shown to be $R_1(x)=f(c)(x-a)$ for some c between a and x.

Inductive Step: Assume the theorem holds for n. Then, for n+1:

$$h(x)=h(a)+h(a)(x-a)+\ldots+\frac{(h^{(n)}(a)}{n!}(x-a)^n + R_n(x) \quad (4)$$

The remainder $R_n(x)$ can be shown as the form:

$$R_n(x)=\frac{h^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad (5)$$

where c between a and x.

By induction, the theorem holds for all n.

Examples of Taylor Series will be given next.

Exponential Function: The Taylor series for $f(x)=e^x$ around a=0 is:

$$e^x = 1 + x = \frac{x2}{2!} + \frac{x3}{3!} + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad (6)$$

Sine Function: The Taylor series for f(x)=sin(x) around a=0 is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!} \quad (7)$$

Cosine Function: The Taylor series for f(x)=cos(x) around a=0 is:

$$cos(x) = 1 - \frac{x2}{2!} + \frac{x4}{4!} + \frac{x6}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{\{2n\}}}{(2n)!} \quad (8)$$

2.2 Applycation

Taylor's Theorem is a powerful tool that allows us to approximate functions with polynomials, making it invaluable in analysis, physics, and engineering. The examples of e^x, sin(x), and cos(x) illustrate its practical applications in approximating common mathematical functions.

One of the most significant applications of Taylor's Theorem is in the approximation of functions, particularly in numerical methods and computer science.

The exponential function e^x can be approximated using its Taylor series expansion around a=0:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = n = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad (9)$$

In many practical situations, we need to compute values of e^x for various x, but calculating the exact value of e^x directly can be inefficient or infeasible, especially for large x. Instead, we can use the Taylor series to get a polynomial approximation that is easier to compute.

We can decide how many terms of the Taylor series to include. For example, we might use the first four terms for a reasonable approximation:

$$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} \quad (10)$$

The actual value of $e^{0.1}$ is approximately 1.1051702. The approximation is very close.

Consider a ball thrown vertically upward with an initial velocity v_0=20m/s. We want to approximate the height h(t) of the ball at small times after launch using the Taylor series expansion around t=0.

Proof: The height of the ball as a function of time is given by:

$$h(t) = v_0 t - \frac{1}{2}gt^2 \quad (11)$$

where g≈9.81m/s^2 is the acceleration due to gravity. We can expand this function around t=0: $h(t) \approx h(0) + h'(0)t + \frac{h''(0)}{2}t^2$. Calculating the derivatives:

$$h(0)=0 \quad (12)$$

$$h'(t) = v_0 - gt, \text{ so } h'(0) = v_0 = 20\text{m/s} \quad (13)$$

$h''(t)=-g$, so $h''(0)=-9.81\text{m}/s^2$ (14)

Substituting into the Taylor series:

$$h(t) \approx 0 + 20t - 29.81t^2 \quad (15)$$

$$h(t)\approx 20t-4.905\,t^2 \quad (16)$$

Approximation for small t, for small t (e.g., t=1s):

$$h(1)\approx 20(1)-4.905\,(1)^2\approx 20-4.905=15.095\text{m}. \quad (17)$$

The Taylor series is employed to expand thermodynamic potentials, such as the Helmholtz free energy or Gibbs free energy, around equilibrium states. This allows physicists to derive important thermodynamic relations and understand phase transitions by analyzing how small changes in temperature or pressure affect the system.

Consider a mass m=1kg located near the surface of the Earth at height h meters. The gravitational potential energy U(h) is given by: U(h)=mgh

where g=9.81m/s^2.

Proof: An approximation for U(h) around h=0 (ground level) could be found, if using the Taylor series: U(h)≈U(0)+U′(0)h

Calculating the derivatives:

$$U(0)=mg\cdot 0=0 \quad (18)$$

$$U'(h)=mg,\ \text{so}\ U'(0)=9.81\text{N} \quad (19)$$

Substituting into the Taylor series:

U(h)≈0+9.81h. For a small height h=2m: U(2)≈9.81(2)=19.62J

In both examples, the Taylor series simplify the functions for small perturbations around a specific point, allowing for easy calculations and approximations of physical quantities in real-world scenarios [7].

Taylor's formula and calculus still have many applications in financial problems.

Consider a European call option on a stock that is currently priced at S_0=100 USD, with a strike price of K=100 USD, an annual risk-free rate of r=5%, volatility σ=20%, and a time to maturity of T=1 year. The Black-Scholes formula for the price of a European call option is given by:

$$C(S_0,K,r,\sigma,T)=S_0 N(d_1)-Ke^{-rT}N(d_2) \quad (20)$$

Where $d_1=\frac{ln(S_0}{/}K)+(r+\sigma^2/2)T\sigma\sqrt{T}$ $d_2=d_1-\sigma\sqrt{T}$

ln(S_0/K)=ln(100/100)=0

$$d_1=\frac{0+(0.05+0.2^2/2)\cdot 1}{0.2\cdot\sqrt{1}}=\frac{0.05+0.02}{0.2}=\frac{0.07}{0.2}=0.35 \quad (21)$$

$$d_2=0.35-0.2=0.15 \quad (22)$$

To approximate the option price, Taylor series expansion can be used around the point where σ=0 [8]. The cumulative distribution function N(d) (the standard normal distribution) can be approximated using a Taylor series:

$$N(d)\approx\frac{1}{2}+\frac{1}{\sqrt{2\pi}}d-\frac{d_3}{6\sqrt{2\pi}}+O(d_5) \quad (23)$$

Using this approximation for d_1 and d_2 :

$$N(0.35)\approx\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\cdot 0.35\approx 0.5+0.139=0.639 \quad (24)$$

$$N(0.15)\approx\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\cdot 0.15\approx 0.5+0.0598=0.5598 \quad (25)$$

Calculating the Call Option Price: Now substitute N(d_1) and N(d_2) back into the Black-Scholes formula:

$$C\approx 100\cdot 0.639-100\cdot e^{-0.05}\cdot 0.5598 \quad (26)$$

$$C\approx 63.9-100\cdot 0.9512\cdot 0.5598\approx 63.9-53.29=10.61\text{USD} \quad (27)$$

Find the Taylor series expansion of the function h(x)= e^x centered at a=0 (Maclaurin series) and determine the radius of convergence [9].

Proof: To find the Taylor series f(x)= e^x of at a=0 , we use the formula:

$$h(a)+h'(a)(x-a)+\frac{h''(a)}{2!}(x-a)^2+\frac{h'''(a)}{3!}(x-a)^3+\ldots+\frac{h^{(n)}(a)}{n!}(x-a)^n+R_n(x) \quad (28)$$

$$f(x)=e^x=f(0)=e^0=1 \quad (29)$$

First derivative: h'(x)= e^x ⇒ h'(0)= e^0 =1

Second derivative: h''(x)= e^x = h''(0)= e^0 =1

Third derivative: h'''(x)= e^x = h'''(0)= e^0 =1

In fact, all derivatives of e^x at x=0 equal 1.

The Taylor series expansion of e^x centered at 0 is:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad (30)$$

The radius of convergence is R=1 , meaning the series converges for all real numbers x [10].

3. Conclusion

In conclusion, this study demonstrates the profound impact of Taylor series and the calculus on various disciplines, especially physics and finance. Taylor series provide a powerful tool for approximating complex functions. Looking ahead, future research could explore combining Taylor series with emerging computational methods such as machine learning and artificial intelligence to further improve predictive capabilities. Additionally, studying the application of Taylor series in other fields, such as biology and economics, could yield valuable insights and broaden the understanding of this fundamental mathematical concept. Ultimately, the continued exploration of Taylor series is expected to open up new avenues for innovation and problem solving in different disciplines.

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