ISSN 2959-6157

Research and Analysis of the Small Disturbance Equation in Subsonic, Transonic, and Supersonic Regimes

Jinglun Li

Faculty of Applied Science and Engineering, University of Toronto, Toronto, Canada
stevejl.li@mail.utoronto.ca

Abstract:

This paper explores the application of the Small Disturbance Equation (SDE) across subsonic, transonic, and supersonic flow regimes. Derived from the Euler and Navier-Stokes equations, the SDE offers an efficient framework for analyzing aerodynamic behaviors, particularly through the utilization of discretization techniques and iterative solving methods executed in Python. The study assesses the accuracy and limitations of the SDE in detailing essential flow characteristics, revealing that while the equation performs effectively in subsonic and transonic flows, it encounters challenges in supersonic regimes. Nonlinear effects such as shock waves significantly hinder its performance at high speeds. Compared with conventional computational fluid dynamics (CFD) methods, the SDE stands out in scenarios where computational efficiency is paramount. However, its limitations in handling high-speed flows must be carefully considered, highlighting the need for further refinement in its application to supersonic dynamics. This analysis suggests that while the SDE is beneficial for certain aerodynamic studies, its scope and utility are constrained by the inherent complexities of high-speed fluid dynamics.

Keywords: Computational fluid dynamics; Small disturbance equations.

1. Introduction

Fluid dynamics plays a pivotal role in aerospace engineering, crucial for enhancing aircraft performance and ensuring safety. The Small Disturbance Equation (SDE), derived from the Euler and Navier-Stokes equations, provides a simplified yet effective model for analyzing fluid behavior, particularly in environments where disturbances are relatively minor compared to the overall flow. This equation is especially beneficial for evaluating thin airfoils in subsonic and transonic regimes [1][2][3]. By simplifying the full potential equations, the SDE facilitates more efficient computational analyses, maintaining essential fluid dynamic characteristics while reducing the computational burden [4][5]. This efficiency is invaluable in the early stages of aerodynamic design, allowing for rapid and reliable assessments without the extensive computational demands of more complex models.

Research Problem: While the SDE proves effective in subsonic and transonic conditions, capturing key flow dynamics such as speed, pressure, and the interaction between fluid and aerodynamic surfaces [6], it encounters significant challenges when extended to supersonic regimes. The primary issue arises from the SDE's inability to handle nonlinear phenomena, notably shock waves [7]. This limitation becomes particularly evident as the flow transitions to higher speeds, where the SDE struggles to maintain accuracy. The inability to predict flow behavior accurately in supersonic conditions suggests a need for caution in applying the SDE to high-speed scenarios, where more sophisticated models may be necessary to capture the complexities of supersonic dynamics accurately [8].

This Paper's Contribution: This paper examines the application and limitations of the Small Disturbance Equation across various flow regimes, with a specific focus on its performance in subsonic, transonic, and supersonic conditions. The study utilizes discretization techniques and iterative solving methods programmed in Python to analyze the SDE's effectiveness in detailing essential aerodynamic behaviors. It demonstrates that while the SDE is advantageous for scenarios demanding computational efficiency, such as in preliminary design phases, its applicability in supersonic flows is restricted due to its inadequate handling of nonlinear effects like shock waves. Through comprehensive analysis, this paper highlights the necessity for further refinement of the SDE or the adoption of more advanced computational fluid dynamics methods when addressing the challenges of supersonic flow dynamics, thus contributing valuable insights for aerodynamic researchers and practitioners.

2. Background

The small disturbance equations (SDE) are derived from 1D Euler equations. Navier-Stokes equations describe the general behaviour of the flow [9]. The equations are illustrated below:

Conservation of Mass/Continuity Equation:

$$\frac{\partial \rho}{\partial t} + div(\rho U) = 0 \quad (1)$$

Conservation of Momentum/Momentum Equation:

$$\frac{\partial(\rho U)}{\partial t} + div(\rho U ? U + pI) = div\tau \quad (2)$$

Conservation of Energy/Energy Equation:

$$\frac{\partial(\rho E)}{\partial t} + div(\rho E U) = div(k\nabla T) + div(\sigma U) \quad (3)$$

1D Euler equations describe the behaviour of 1D compressible, inviscid flow and can be derived from the equations above with following relations:

$$p = (\gamma - 1)\rho e \quad (4)$$

$$c^2 = \gamma p / \rho \quad (5)$$

The 1D Euler Equations can be obtained:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = 0 \quad (6)$$

In the case when vorticity $\nabla \times v$ is zero, $v = \nabla\varphi$ in which φ represents the flow potential. The continuity equation can be represented as below:

$$\frac{\partial \rho}{\partial t} + div(\rho \nabla \varphi) = 0 \quad (7)$$

For steady flow, the time derivative disappears and we have the relation of density with a single unknown:

$$\frac{\rho}{\rho_0} = (1 - \frac{|\nabla\varphi|^2}{2H_0})^{\frac{1}{\gamma-1}} \quad (8)$$

Substitute equation (8) into (7), the steady full potential equation in non-conservative form can be obtained:

$$\left(1 - M_x^2\right)\varphi_{xx} + \left(1 - M_y^2\right)\varphi_{yy} + \left(1 - M_z^2\right)\varphi_{zz} - 2M_x M_y \varphi_{xy} - 2M_x M_z \varphi_{xz} - 2M_y M_z \varphi_{yz} = 0 \quad (9)$$

$M_x = \frac{\varphi_x}{c}, M_y = \frac{\varphi_y}{c}, M_z = \frac{\varphi_z}{c}, M^2 = \frac{|\nabla\varphi|^2}{c^2}$ with wall BC

$$v_n = \frac{\partial \varphi}{\partial n} = 0 \quad (10)$$

$$c^2 = (\gamma - 1)\left[H_0 - \frac{1}{2}|\nabla\varphi|^2\right] \quad (11)$$

The small disturbance equation is a simplification of the full potential equations in case of thin obstacles such as thin airfoils [10]. In this paper, the equations are restricted to 2-dimension. The perturbation is considered to change uniform flow in x-direction with velocity of magnitude U_∞. The potential can be represented as below:

$$\varphi = U_\infty(x + \varphi) \quad (12)$$

Since this paper focus on 2D situations with uniform flow in x-direction, velocities in x and y directions can be expressed as:

$$u = U_\infty(1 + \varphi_x) \quad (13)$$

$$v = U_\infty \varphi_y \quad (14)$$

In this case, the full potential equation becomes:

$$\left(1 - M_x^2\right)\varphi_{xx} + \varphi_{yy} = 0 \quad (15)$$

In some cases when M∞ is small can be considered as incompressible, Mx can be simplified as M∞ to reduce the complexity of computation.

The wall boundary condition (BC) becomes:

$$v = (U_\infty + u) f'(x) \approx U_\infty f'(x) \quad (16)$$

Research Object: The objective of this paper is to analyze the small disturbance equation (SDE) within the context of subsonic, transonic, and supersonic flow regimes. The Small Disturbance Equation is used as a simplified model of Navier-Stokes equations for studying the behavior of fluid flows. [2] In this paper, analysis of flow effects such as shock wave will be carried out and detailed flow charts with visualizations of the streamlines will be presented. The goal is to assess the performance of SDE based on the behaviour in different flow speeds and provide guidance on the use of SDE for various aerodynamic design and analysis applications.

3. Research Method

3.1 Computational Approach

In this paper, the computational approach for solving the small disturbance equation (SDE) is based on discretization methods of derivatives. The full potential equation derived in 2.1 has 2 second-order derivates of velocity potential in x and y direction. These derivates can be discretized using central difference scheme as below:

$$\varphi_{xx} = \frac{1}{?x^2}(\varphi_{i-1,j} - 2\varphi_{i,j} + \varphi_{i+1,j}) \quad (17)$$

$$\varphi_{yy} = \frac{1}{?y^2}(\varphi_{i,j-1} - 2\varphi_{i,j} + \varphi_{i,j+1}) \quad (18)$$

To simplify the calculations, assume $?x = ?y = h$. The discretization form of full potential equation can be written as:

$$\left(1 - M_x^2\right)(\varphi_{i-1,j} - 2\varphi_{i,j} + \varphi_{i+1,j}) + (\varphi_{i,j-1} - 2\varphi_{i,j} + \varphi_{i,j+1}) = 0$$

(19)

Rearrange the equation so that $\varphi_{i,j}$ can be expressed as:

$$\varphi_{i,j}=\frac{\left(1-M_x^2\right)\left(\varphi_{i-1,j}+\varphi_{i+1,j}\right)+(\varphi_{i,j-1}+\varphi_{i,j+1})}{4-2M_x^2} \quad (20)$$

When it comes to the supersonic case, the wave equation below plays an important role:

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \quad (21)$$

As the full potential equation in 2.1 can be written in the form of wave equation, the stability verification method for wave equation is validate for full potential equation. Based on the calculation, the central difference method used for subsonic flow is unstable for the supersonic case. Based on the wave equation, the supersonic case uses finite difference approximations:

$$\left(1-M_x^2\right)(\varphi_{i,j}-2\varphi_{i-1,j}+\varphi_{i-2,j})+(\varphi_{i,j-1}-2\varphi_{i,j}+\varphi_{i,j+1})=0 \quad (22)$$

$$\varphi_{i,j}=\frac{\left(1-M_x^2\right)\left(-2\varphi_{i-1,j}+\varphi_{i-2,j}\right)+(\varphi_{i,j-1}+\varphi_{i,j+1})}{1+M_x^2} \quad (23)$$

For transonic case, K is introduced as:

$$K=1-M_x^2-\left(1-\gamma\right)M_x^2\varphi_x \quad (24)$$

The discretization form of K is written as:

$$K_{i,j}=1-M_x^2-\left(1-\gamma\right)M_x^2\frac{\varphi_{i+1,j}-\varphi_{i-1,j}}{2} \quad (25)$$

Combining the transonic and supersonic full potential equations get:

$$\mu_{i.j}K_{i-1,j}\left(\varphi_{i,j}-2\varphi_{i-1,j}+\varphi_{i-2,j}\right)+\left(\varphi_{i,j-1}-2\varphi_{i,j}+\varphi_{i,j+1}\right)+\left(1-\mu_{i.j}\right)K_{i,j}\left(\varphi_{i-1,j}-2\varphi_{i,j}+\varphi_{i+1,j}\right)=0 \quad (26)$$

in which $\mu_{i.j}$ equals 0 when Ma<1 and 1 when Ma>1.

Let $A_{i,j}=\mu_{i.j}K_{i-1,j}$ and $B_{i,j}=(1-\mu_{i.j})K_{i,j}$, $\varphi_{i,j}$ of transonic situation is:

$$\varphi_{i,j}=\frac{A_{i,j}\varphi_{i-2,j}+(B_{i,j}-2A_{i,j})\varphi_{i-1,j}+B_{i,j}\varphi_{i+1,j}+\varphi_{i,j-1}+\varphi_{i,j+1}}{-A_{i,j}+2+2B_{i,j}} \quad (27)$$

The code is created in Python using iteration method. It applies an iterative method to solve the linear algebra equations of the flow potential. The key function of the codes begins with an initial guess for flow potential and updates it through iterative solving of formula dp equals discretization form of $\varphi_{i,j}$ minus newphi[i,j]. A second function calculates the error during the analysis. It returns its calculations every 100 iterations and can check if the scheme converges. The third function decomposes the velocity field into x and y components, which is crucial to create streamline plots. The last function is related to boundary conditions of the bottom and will be discussed in section 3.2.

3.2 Boundary Conditions and Flow Situations

The 2D flow region studied in this paper will be restricted in domain $x\in\left[0,3\right]$ and $y\in\left[0,1\right]$. The left and right boundaries are based on the values of the flow in infinity condition. Left boundary is the inflow and right boundary is the outflow. The top boundary is the same as the left and right boundaries. It is an imaginary limit that defines the region the python code should calculate. The bottom boundary is a Neumann boundary. It has a bump in $x\in\left[1,2\right]$ and can be expressed based on its slope function denoted as g(x):

$$g\left(x\right)=\begin{cases}\beta cos\left(\pi\left(x-1\right)\right), x\in\left[1,2\right]\\ 0, otherwise\end{cases} \quad (28)$$

In this paper, $\beta=0.2$ is selected.

To discretize the Newmann boundary condition on the bottom, an imaginary point $\varphi_{i,j-1}$ is created:

$$g_{i,j}=\frac{\varphi_{i,j+1}-\varphi_{i,j-1}}{2h} \quad (29)$$

As a result, we can express the imaginary point $\varphi_{i,j-1}$ in the form of $g_{i,j}$ and $\varphi_{i,j+1}$:

$$\varphi_{i,j-1}=\varphi_{i,j+1}-2hg_{i,j} \quad (30)$$

Substitute equation 30 to the equations of discretization form of $\varphi_{i,j}$ in section 3.1, the discretization form of $\varphi_{i,j}$ on bottom boundary is derived:

Subsonic:

$$\varphi_{i,j}=\frac{\left(1-M_x^2\right)\left(\varphi_{i-1,j}+\varphi_{i+1,j}\right)+2\varphi_{i,j+1}-2hg_{i,j}}{4-2M_x^2} \quad (31)$$

Supersonic:

$$\varphi_{i,j}=\frac{\left(1-M_x^2\right)\left(-2\varphi_{i-1,j}+\varphi_{i-2,j}\right)+2\varphi_{i,j+1}-2hg_{i,j}}{1+M_x^2} \quad (32)$$

Transonic:

$$\varphi_{i,j}=\frac{A_{i,j}\varphi_{i-2,j}+(B_{i,j}-2A_{i,j})\varphi_{i-1,j}+B_{i,j}\varphi_{i+1,j}+2\varphi_{i,j+1}-2hg_{i,j}}{-A_{i,j}+2+2B_{i,j}} \quad (33)$$

For subsonic case, we choose Ma = 0.3 as our Mach number. Ma = 0.95 is selected for the transonic case so that the flow can accelerate when going over the nozzle. The supersonic case chooses flow speed of Ma = 2.5.

3 Results and Discussion

4.1 Simulation results

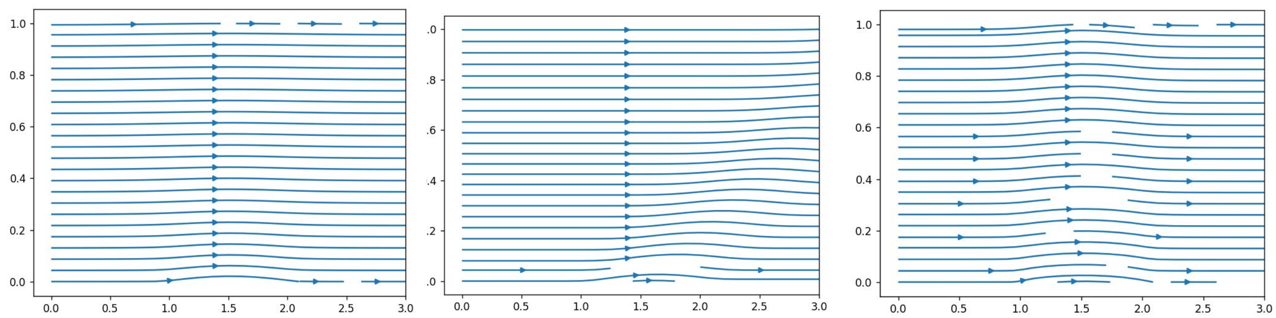

Fig. 1 Streamlines in subsonic, supersonic and transonic flow regimes (Photo credit: Original).

As show in the Fig. 1. The streamline plots show different flow behavior based on the Mach number. In the subsonic regime, the streamlines indicates that the flow is smooth and does not exhibit large disturbances or discontinuities. The uniform spacing of streamlines illustrates small pressure changes, and the flow is mostly attached to the surface. The supersonic regime shows distinct changes compared to subsonic flow. The streamlines exhibit significant deflections and appear more compressed. The appearance of a shock wave is illustrated by the turning of streamlines going from the leading edge of the bump to the top right. The streamline plot of transonic flow shows a mixed flow behavior. The flow accelerates across the bump, especially near the center. There are discontinuities of streamlines near the center, indicating the occurrence of compression and expansion waves.

4.2 Error Analysis and Future Improvements

For all three flow regimes, error decreases as the number of iterations increases. Supersonic scheme has a different behaviour as other two schemes. Despite the initial reduction of error until 100 iterations, the error remains unchanged afterwards. This indicates that the supersonic scheme may struggle with nonlinear supersonic flow behaviour such as shock waves. Both Subsonic and transonic scheme shows the convergence behavior. The error decreases rapidly with the increasing iteration steps and follows an exponential decay pattern. The error reduction of transonic scheme is slower than that of subsonic scheme, reflecting the complexity of transonic flow. As show in the table 1.

Table 1. Error under different flow regimes with number of iteration steps.

Number of Iterations	M_∞	0.3	2.5	0.95
0		4.25520×10^{-5}	1.40047×10^{-4}	4.33907×10^{-5}
100		1.39127×10^{-5}	5.26928×10^{-6}	2.27165×10^{-5}
200		4.93359×10^{-6}	5.26928×10^{-6}	1.05248×10^{-5}
300		1.73264×10^{-6}	5.26928×10^{-6}	4.93265×10^{-6}
400		6.10015×10^{-7}	5.26928×10^{-6}	2.32813×10^{-6}
500		2.17196×10^{-7}	5.26928×10^{-6}	1.10341×10^{-6}

4 Conclusion

This study rigorously evaluates the Small Disturbance Equation (SDE) for its applicability across various flow regimes, underscoring its efficiency and reasonable accuracy in subsonic and transonic conditions. However, the investigation reveals significant limitations in its performance when extended to supersonic flows, primarily due to the inability of the SDE to effectively handle nonlinear phenomena such as shock waves. These findings illuminate the SDE's dual character: while it serves as a robust tool in environments where computational speed is prioritized, its use in high-speed aerodynamic analyses demands caution due to potential discrepancies between the model predictions and actual flow behaviors.

Given the observed limitations of the SDE in supersonic conditions, future research should focus on refining the equation or developing new methodologies that enhance its applicability and accuracy in high-speed flow regimes. This could involve integrating the SDE with more complex computational fluid dynamics (CFD) models that are better equipped to handle the intricacies of nonlinear effects. Further, there is an opportunity to explore the integration of machine learning techniques with traditional fluid dynamics models to predict and correct the limitations inherent in current methodologies. Such advancements could significantly improve the predictive capabilities of simulation tools, ultimately leading to more reliable and efficient designs in aerospace engineering and beyond. These future directions not only aim to expand the scope of the SDE but also enhance its practical relevance in the field of fluid dynamics.

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