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# The Application of Calculus in Physics

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**Abstract:**
The derivative is a fundamental tool that quantifies the sensitivity of change for a function's output with respect to its input. The derivative of a function for a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point. The application of calculus, including derivative and integral, is widely used in solving problems in the subject of Physics Kinematics. This paper uses derivative to calculate problems involving position, velocity, and acceleration, as well as using integral to solve problems about continuous forces and forces in spring. In calculus, the main idea of derivative is the segmentation process which will be carried on indefinitely and the local range is infinitely smaller. The main idea of integral is the sum of the results of all the infinite differential elements. These ideas can be used to simplify the kinematics question and accurate the final answers.

**Keywords:** Calculus; application; Physics.

## 1. Introduction

Differentiation, as a powerful tool in calculus with the main idea of processing infinitely small segments for infinitely small local range [1]. Integral uses the main idea of the sum of the results of differentiation which can be used to calculate unknown variables including position, velocity, acceleration, and different forces in Physics Kinematics [2].

In past studies, derivative turns out to be useful tools in various fields, including economics, engineering, and biology. Derivative in its application for economics is done with three important functions, namely cost, income function and maximum profit function. Three are three steps: firstly, determining a mathematical model of economic problems, then completing the mathematical model and finally interpreting the results in solving the problem [3]. For the details of creep phenomena in engineering fields, fractional derivative model is regarded as an effective approach that can not only well reflect the process of nonlinear gradual change at primary stage but also present accurately at steady stage and accelerated stage under high stress [4]. Moreover, the idea of fractional derivative can be applied to a wide range of models in systems biology which becomes a great step forward to analyze and study the models in more details and calculate some numerical approximate solutions [5].

In this paper, the concept of derivative and integral will be demonstrated both separately and mixed in solving problems about Physics Kinematics. The fundamental relationships between position, velocity, acceleration, and how they can be expressed using derivatives will be demonstrated. This will include the key concept that acceleration is the second derivative of position, and how this allows for the analysis and prediction of object motion using calculus. Additionally, the application of integrals in calculating the work done by a force acting on an object will be discussed. The integral represents the cumulative effect of the force as the object moves from one position to another which is a powerful tool for understanding the energy transfer in various physical systems [6].

## 2. Application of Calculus

### 2.1 The Application of Calculus in Kinematics

#### 2.1.1 Derivative and Velocity

In Kinematics, derivative can be used as a tool in calculus to determine the velocity of objects with a certain direction. The concept of velocity can be described as the rate of change for an object's position with respect to time. Let the function $p(t)$ represent the relationship between the position and time. $\frac{dp}{dt}$ is the derivative of $p(t)$ which can be seen as the velocity of the object $v(t)$. The velocity computed from the derivative is seen as the instantaneous velocity at a given moment [7].

Example 1:

A cyclist is traveling along a straight path, and his position at time $t$ in seconds is given by the function $p(t)=5t^3-3t^2-2t$ meters.

a) What is the cyclist's velocity at $t=4$ seconds?
b) When is the cyclist momentarily at rest?

Proof: The function of velocity at the time t is shown as $v(t)=\frac{dp}{dt}=p'(t)=15t^2-6t-2$. The cyclist's velocity at $t=4s$ is $p'(4)=15(4)^2-6\times4-2=214 meters/seconds$. The time of the cyclist is momentarily at rest is simply when his velocity is $0 meters/seconds$. Therefore, computing this equation $p'(t)=15t^2-6t-2=0$, so that according to the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, when $x=\frac{3+\sqrt{39}}{15}$ and $x=\frac{3-\sqrt{39}}{15}$ (which is negative). Thus, when t=$\frac{3+\sqrt{39}}{15}s$, the cyclist is momentarily at rest.

**2.1.2 Derivative and Acceleration**

Acceleration, in calculus, can be seen as the rate of change for the velocity with respect to time. Let the function v(t) be the function of velocity. Therefore, $v'(t)(p''(t))$ turns out to be the acceleration of the object, $a(t)$, which equals to $\frac{dv}{dt} and \frac{d^2p}{dt^2}$ [8].

Example 2:

The position of a particle moving along a straight line at any time t is given by the function $p(t)=2t^3-5t^2-10$.

a) What is the acceleration of the particle when $t$ is 4?
b) At what time is the acceleration of the particle zero?

Proof: From the function $p(x)$, its second derivative can be used to compute the function of acceleration $a(t)=12t-10$. Therefore, the acceleration of the particle when $t$ is 4 is $38 meters/second^2$. The acceleration of the particle is zero when $a(t)=0$, t is equal to $\frac{5}{6}$ *seconds.*

**2.1.3 Derivative in Combination of Velocity and Acceleration**

With the above concept of the relationship between derivative, velocity and acceleration, derivative can also be used in determining specific unknown values along with the integral.

Example 3: A particle moving along a straight line has velocity $v_0$ at t=0 with decelerated motion. Since then, the directions of its acceleration and velocity are opposite at any time, and is proportional to the velocity $a=-kv$. How long does this particle need to come to a complete stop? How does this particle travel?

Proof: According to the question, the relationship between acceleration and velocity is $a=-kv$. Then derivative and integral with time and initial velocity can be used to find further relationship.

$$\frac{dv}{dt}=a \tag{1}$$

$$\frac{dv}{dt}=-kv \tag{2}$$

$$\frac{dv}{v}=-kdt \tag{3}$$

$$\int_{v_0}^{v}\frac{dv}{v}=-k\int_0^t dt \tag{4}$$

$$lnv\,\Big|_{v_0}^{v}=-kt \tag{5}$$

$$ln\frac{v}{v_0}=-kt \tag{6}$$

$$v=v_0\times e^{-kt} \tag{7}$$

According to the graph of $e^{-kt}$, v could not reach 0 ($e^{-kt}$ has horizontal asymptote x-axis). Therefore, the particle could not come to a complete stop, the time t need is $\infty$.

Let x represent the total displacement the particle travels.

$$x=\int_0^\infty vdt \tag{8}$$

$$x=\int_0^\infty v_0\times e^{-kt}dt \tag{9}$$

Let -kt be u

$$u=-kt \tag{10}$$

$$\frac{du}{dt}=-k \tag{11}$$

$$dt=\frac{du}{-k} \tag{12}$$

$$\int_0^\infty v_0\times e^u\frac{du}{-k}=\frac{v_0}{-k}\int_0^\infty e^u du=-\frac{v_0}{k}\int_0^\infty e^{-kx}d(-kx)=-\frac{V_0}{k}\cdot e^{-kt}\Big|_0^\infty =\frac{v_0}{k} \tag{13}$$

From the above, the total displacement of the particle turns out to be $\frac{v_0}{k}$.

## 2.2 Application of Integral in Work and Force

Assume a force $F(x)$ is working continuously on an object for a period of time t, what is the work done during the process of the object moving from $x=a$ to $x=b$? (illustration shown in Fig 1)

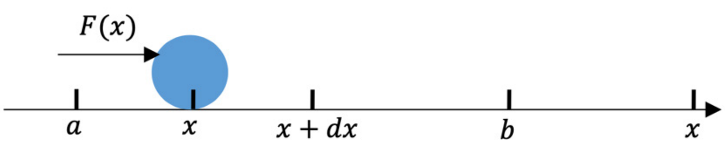


**Fig 1. Object moving from $d=a$ to $d=t$**

Mathematically, if a force F(x) is acting on an object as it moves from position x=a to position x=b, the work done can be calculated with integrals. According to the infinitesimal method, on the interval $[x, x+dx]$, $dW = F(x)dx$, $W=\int_a^b dW = \int_a^b F(x)dx$ [9].

This integral represents the area under the curve of the force function F(x) over the interval [a, b]. The work done is the sum of the force acting on the object as it moves through this interval [10].

Example 4: Assume a particle is x meters away from the origin, what's the work W done by the particle moving from x=1 to x=3 with force $F(x)=x^2+2x$?

Proof: Using the equation from the question, $W=\int_a^b F(x)dx = \int_1^3 x^2+2xdx = \left[\frac{1}{3}x^3+x^2\right]_1^3 = 16\frac{2}{3}J$

Example 5: The force needed to stretch the spring from 10cm to 15cm is 40N. (See illustration of spring stretched from Fig 2 to Fig 3) How much work is needed to overcome the elastic recovery force and extend 3cm more from where it's 15 cm?

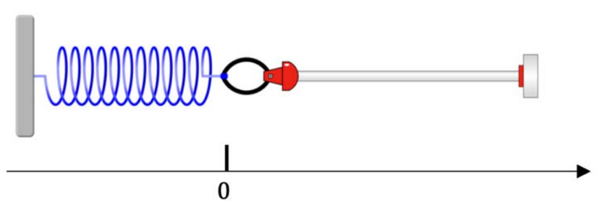


**Fig 2. Spring at d=0**

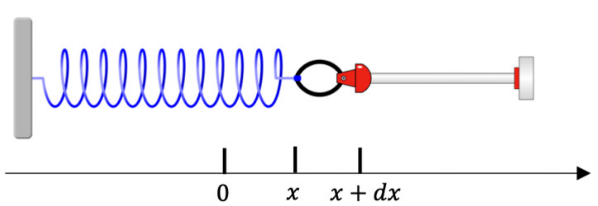


**Fig 3. Spring at d=x**

Proof: According to Hooke's Law, $F(x)=kx$, in which k is the springs constant. When stretching from 0.10m to 0.15m, $F(0.05m)=k\times 40N$, $k=800$. The general function is $F(x)=800x$. $W=\int_{0.05}^{0.08} 800xdx = \left[400x^2\right]_{0.05}^{0.08}$ =1.56J.

Example 6: Using a hammer to hit the iron nail into the wooden block. Assume the force needed and the depth the iron nail went going into the block are in direct proportion. After hitting it into the block once, the depth is 1cm. If the hammer is doing the same amount of work for each hit, what's the depth of the iron nail after the second hit?

Proof: The force and depth are in direct proportion, $F(x)=kx$. The work done for the first hit is $W=\int_0^1 kxdx = \frac{k}{2}$. According to the work done by the second hit,

$$W=\frac{k}{2}=\int_1^h kxdx = \frac{k}{2}\left(h^2-1\right) \tag{14}$$

$$h^2=2 \tag{15}$$

$$h=\sqrt{2} \tag{16}$$

Now, the final depth is (1+$\sqrt{2}$)cm.

Example 7: Assume the author lifts water from deep wells. The bucket weighs 4 kg, the cable weighs 2 kg per meter, and the author uses the cable to lift water from a well 30 meters deep. The bucket initially contains 40 kg of water, and rises at a constant speed of 2 meters/second, and the water in the bucket flows out of the hole in the wall at a rate of 0.2 kg/second. How much work does the bucket lift to the wellhead mine?

Proof: There are three works needed to be considered for the total work. $W_1$ is the work done needed to pull up the cable. $W_2$ is the work done needed to pull up the bucket. $W_3$ is the work done needed to pull up the water.

$$dW_1=(30-x)\times 2kg\times gdx \tag{17}$$

$$W_1=\int_0^{30} 2g(30-x)dx = 9000J \tag{18}$$

$$W_2=m_2gh=4g\times 30=1200J \tag{19}$$

From the question, the time needed to raise the water is $\frac{30m}{\frac{2m}{s}}=15s$. The water weighs $(40-0.2t)g$ N at time T.

On the interval $[t, t+dt]$, the distance to raise the water is $vdt = 2dt$.

$$dW_3 = m_3 g dx = (40 - 0.2t) g \times 2dt \tag{20}$$

$$W_3 = \int_0^{15} 2(40 - 0.2t) g dt = 11550J \tag{21}$$

$Totalwork: W_t = W_1 + W_2 + W_3 = 9000J + 1200J + 11550J = 21750J$

## 3. Conclusion

This paper has demonstrated the application of derivative and integral in the subject of physics kinematics. The direct connection between acceleration and the second derivative of position is widely used in modeling of object motion using calculus as different tools in subjects such as economics, engineering and biology. This understanding enables the accurate determination of an object's velocity and acceleration given its position function. Furthermore, the application of integrals in calculating the work done by a force acting on an object as well as the work done in spring using Hooke's Law has been demonstrated. By using the power of derivatives and integrals, researchers can optimize the design and efficiency of a wide range of systems and devices of different fields. Looking ahead, the application of calculus in kinematics is poised to expand even further in the future. As technology advances and the need for precise motion analysis and control grows, the application of these mathematical tools will continue to be crucial. Future research may explore the integration of calculus with emerging technologies, such as artificial intelligence and machine learning, to enhance the predictive capabilities of technologies. Additionally, the exploration of calculus in the context of more complex and multidimensional motion patterns, as well as its potential applications in the study of biological and ecological systems can be crucial for future investigation.

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