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# Dirichlet Integral and Its Generalization by Different Methods

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**Abstract:**

Dirichlet integral, as the integration of sine function over $x$ from positive infinity to 0, or negative infinity, is known as one of the most important concepts in mathematical analysis. In probability and statistics, Dirichlet integral is used in calculating expectation and variance. In physics, Dirichlet integral is used in describing the movement of a charged particle in specific electromagnetic fields, and the wave function of particles in quantum physics. In computer science, Dirichlet integral is used in the optimization of neural network. In economics and management, Dirichlet integral is used in the optimization of production and distribution, the estimation of the risk and return in investment. In this paper, Dirichlet Integral and its generalization are studied. This paper provides several methods including residue theorem and Feynman's trick to determine the value of Dirichlet integral. Meanwhile, this paper extends Dirichlet integral to the case of $n$-th power and deduces a general solution by gamma function for the integral.

**Keywords:** Dirichlet integral; residue theorem; Feynman's trick; gamma function.

## 1. Introduction

When integration was first introduced in calculus, some technics, like integration by substitution and integration by parts are also introduced to find the antiderivative of some proper integrals, and the value. However, integrals having antiderivative in the form of elementary functions are the minority. Most of the integrals do not have antiderivative, which are called transcendental functions. Some of the integrals, which have transcendental antiderivative, usually look simple in form, but are difficult, and even impossible to solve. One famous example is the Dirichlet integral:

$$\int_0^\infty \frac{sinx}{x}dx = \frac{\pi}{2}. \tag{1}$$

In 2005, Paul found a solution to derive the value of Dirichlet integral and Fresnel integral via iterated integration [1]. In 2007, Zhu provided a method to determine the convergence of Dirichlet integral [2]. In 2014, Wang, Chang and Yu introduced functions of complex variables, and 3 ways to find the value of Dirichlet integral by integration transformation [3]. In 2015, Duan utilized the knowledge related to probability and statistics, deducing a new formula for the multi-integral form of Dirichlet integral [4]. In 2023, Listo provided examples solving problems in calculus using probability. One of them is the $n$-dimensional Dirichlet integral [5].

This paper will review some famous methods to determine the value of Dirichlet integral, which are residue theorem, and Feynman's trick. Furthermore, this paper investigates an extension of the Dirichlet integral, which is called the $n$-th powered Dirichlet integral:

$$\int_0^\infty \frac{sin^n x}{x^n}dx. \tag{2}$$

It gives a general solution to the extended integral, depending on the parity of $n$, via the involvement of gamma function, multi-variable integration, and residue theorem.

## 2. The Calculation of Dirichlet Integral

### 2.1 Integration by Residue Theorem

By residue theorem [6],

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i\left(\sum_{k=1}^{n} Res[f, z_k] + \frac{1}{2}\sum_{k=1}^{m} Res[f, x_k]\right) \tag{3}$$

where $z_k$ are the singularities of $f(z)$ in the upper half plane, and $x_k$ are the singularities of $f(z)$ on the real axis of order 1. Since the only singularity of $\frac{e^{iz}}{z}$ is $z=0$,

$$\int_{-\infty}^{+\infty} \frac{e^{iz}}{z}dz = \pi i \cdot Res[f, 0] = \pi i. \tag{4}$$

Therefore, comparing the imaginary part on both sides of (2) [7],

$$\int_0^\infty \frac{sinx}{x}dx = \frac{1}{2}Im\left[\int_{-\infty}^{+\infty}\frac{e^{iz}}{z}dz\right] = \frac{\pi}{2}. \tag{5}$$

## 2.2 Integration by Feynman's Trick

Via Feynman's trick [8], let $I(t)$ denote

$$I(t)=\int_0^{\infty}\frac{sinx}{x}e^{-tx}dx. \tag{6}$$

Then, since $\frac{sinx}{x}e^{-tx}$ obviously converges and is continuous when $0\leqslant t\leqslant+\infty$, by taking the partial derivatives with respect to $t$ in (3),

$$\frac{\partial}{\partial t}I(t)=\frac{\partial}{\partial t}\int_0^{\infty}\frac{sinx}{x}e^{-tx}dx=\int_0^{\infty}\frac{\partial}{\partial t}\frac{sinx}{x}e^{-tx}dx$$
$$=-\int_0^{\infty}e^{-tx}sinxdx. \tag{7}$$

Meanwhile, via integration by parts,

$$\int_0^{\infty}e^{-tx}sinxdx=\left[-e^{-tx}cosx\right]_0^{\infty}-\int_0^{\infty}te^{-tx}cosxdx$$
$$=1-\left(\left[te^{-tx}sinx\right]_0^{\infty}-\int-t^2e^{-tx}sinxdx\right)=1-t^2\int_0^{\infty}e^{-tx}sinxdx. \tag{8}$$

Hence, equation (4) can be rewritten as

$$\frac{\partial}{\partial t}I(t)=-\int_0^{\infty}e^{-tx}sinxdx=-\frac{1}{1+t^2}. \tag{9}$$

Thus, equation (3) is equivalent to

$$I(t)=\int-\frac{1}{1+t^2}dt=-arctant+C. \tag{10}$$

Since

$$\lim_{t\to+\infty}I(t)=\lim_{t\to+\infty}\int_0^{\infty}\frac{sinx}{x}e^{-tx}dx=\lim_{t\to+\infty}-arctant+C=0, \tag{11}$$

$C=\frac{\pi}{2}$. Therefore, $\int_0^{\infty}\frac{sinx}{x}dx=I(0)=-arctan0+\frac{\pi}{2}=\frac{\pi}{2}$.

# 3. The Generalization of Dirichlet Integral.

# References Denote that

$$I_n=\int_0^{\infty}\frac{sin^n x}{x^n}dx. \tag{12}$$

The main obstacle is the $\frac{1}{x^n}$, which will be dealt with by gamma function in this paper [9]. Let

$$\Gamma=\int_0^{\infty}t^{n-1}e^{-xt}dt. \tag{13}$$

Then via substituting by $u=xt$ implies $dt=\frac{1}{x}du$,

$$\Gamma=\int_0^{\infty}t^{n-1}e^{-xt}dt=\int_0^{\infty}\frac{u^{n-1}}{x^{n-1}}e^{-u}\cdot\frac{1}{x}du=\frac{1}{x^n}\int_0^{\infty}u^{n-1}e^{-u}du$$
$$=\frac{1}{x^n}\Gamma(n). \tag{14}$$

Thus, the $\frac{1}{x^n}$ is replaced by $\frac{\Gamma}{\Gamma(n)}$, and

$$I_n=\int_0^{\infty}sin^n x\cdot\frac{1}{x^n}dx=\int_0^{\infty}sin^n x\cdot\frac{1}{\Gamma(n)}\int_0^{\infty}t^{n-1}e^{-xt}dtdx$$
$$=\frac{1}{\Gamma(n)}\int_0^{\infty}t^{n-1}\int_0^{\infty}e^{-xt}sin^n xdxdt. \tag{15}$$

Via integration by parts,

$$J_n=\int_0^{\infty}e^{-xt}sin^n xdx=\int_0^{\infty}\frac{e^{-xt}}{t^2}\left[n(n-1)sin^{n-2}x-n^2sin^n x\right]dx$$
$$=\frac{n(n-1)}{t^2}\int_0^{\infty}e^{-xt}sin^{n-2}xdx-\frac{n^2}{t^2}\int_0^{\infty}e^{-xt}sin^n xdx$$
$$=\frac{n(n-1)}{t^2}J_{n-2}-\frac{n^2}{t^2}J_n. \tag{16}$$

Thus, the iteration equation of $J_n$ is deduced:

$$J_n=\frac{1}{1+\frac{n^2}{t^2}}\cdot\frac{n(n-1)}{t^2}J_{n-2}=\frac{n(n-1)}{n^2+t^2}J_{n-2}. \tag{17}$$

The next step is to solve $J_n$.

In the case that $n$ is even, where $n=2k,k\in\mathbb{N}_+$, since

$$J_0=\int_0^{\infty}e^{-xt}dx=\left[-\frac{1}{t}e^{-xt}\right]_0^{\infty}=\frac{1}{t},$$

$$J_n=\frac{J_n}{J_{n-2}}\cdot\frac{J_{n-2}}{J_{n-4}}\cdots\cdots\frac{J_2}{J_0}\cdot J_0=\frac{n(n-1)}{n^2+t^2}\cdot\frac{(n-2)(n-3)}{(n-2)^2+t^2}$$
$$\cdots\cdots\frac{2\cdot1}{2^2+t^2}\cdot\frac{1}{t}=\frac{n!}{(n^2+t^2)\left[(n-2)^2+t^2\right]\cdots\left[2^2+t^2\right]t}, \tag{18}$$

and

$$I_n=\frac{1}{\Gamma(n)}\int_0^{\infty}t^{n-1}J_ndt=\int_0^{\infty}\frac{nt^{n-2}}{(2^2+t^2)(4^2+t^2)\cdots(n^2+t^2)}dt. \tag{19}$$

Denote that $P_n(t)=\frac{nt^{n-2}}{(2^2+t^2)(4^2+t^2)\cdots(n^2+t^2)}$. Since the singularities of $P_n$ in the upper half of the plane are simple poles $2mi(m=1,2,\cdots,k)[10]$, the residue of $P_n$ at one of the poles will be

$$Res\left[P_n,2mi\right]=(z-2mi)P_n(z)|_{z=2mi}$$
$$=\frac{2k\cdot(2mi)^{2k-2}}{4mi\left[2^2-(2m)^2\right]\cdots\left[(2m-2)^2-(2m)^2\right]\cdots\left[(2k)^2-(2m)^2\right]}$$
$$=\frac{2k\cdot(2m)^{2k-2}\cdot i^{2k-2}}{4mi\prod_{r=1}^{m-1}\left[4r^2-4m^2\right]\prod_{r=m+1}^{k}\left[4r^2-4m^2\right]} \tag{20}$$
$$=(-1)^m\frac{(mi)^{2k-1}k}{(k-m)!(k+m)!}.$$

Therefore, when $n$ is even, where $n=2k, k\in\mathbb{N}_+$,

$$\int_0^\infty \frac{sin^n x}{x^n}dx = \pi i \sum_{m=1}^{k} Res\left[P_n, 2mi\right] = \pi\sum_{m=1}^{k}(-1)^{m+k}\frac{m^{2k-1}k}{(k-m)!(k+m)!}, \tag{21}$$

In the case that $n$ is odd, where $n=2k-1, k\in\mathbb{N}_+$, since

$$J_1=\int_0^\infty e^{-xt}sinxdx=\frac{1}{1+t^2},$$

$$J_n=\frac{J_n}{J_{n-2}}\cdot\frac{J_{n-2}}{J_{n-4}}\cdot\cdots\cdot\frac{J_3}{J_1}\cdot J_1=\frac{n(n-1)}{n^2+t^2}\cdot$$

$$\frac{(n-2)(n-3)}{(n-2)^2+t^2}\cdot\cdots\cdot\frac{3\cdot 2}{3^2+t^2}\cdot\frac{1}{1+t^2} \tag{22}$$

$$=\frac{n!}{\left(n^2+t^2\right)\left[(n-2)^2+t^2\right]\cdots\left(1+t^2\right)},$$

and

$$I_n=\frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}J_n dt=\int_0^\infty\frac{nt^{n-1}}{\left(1^2+t^2\right)\left(3^2+t^2\right)\cdots\left(n^2+t^2\right)}dt. \tag{23}$$

Denote that $P_n(t)=\frac{nt^{n-1}}{\left(1+t^2\right)\left(3^2+t^2\right)\cdots\left(n^2+t^2\right)}$. Since the singularities of $P_n$ in the upper half of the plane are simple poles $(2m-1)i\,(m=1,2,\cdots k)$[10], the residue of $P_n$ at one of the poles will be

$$\begin{aligned} Res\left[P_n,(2m-1)i\right]&=\left[z-(2m-1)i\right]P_n(z)|_{z=(2m-1)i}\\ &=\frac{(2k-1)[(2m-1)i]^{2k-2}}{2(2m-1)i\left[1-(2m-1)^2\right]\cdots\left[(2m-3)^2-(2m-1)^2\right]\cdots\left[(2k-1)^2-(2m-1)^2\right]}\\ &=\frac{(2k-1)(2m-1)^{2k-2}i^{2k-2}}{2(2m-1)i\prod_{r=1}^{m-1}\left[(2r-1)^2-(2m-1)^2\right]\prod_{r=m+1}^{k}\left[(2r-1)^2-(2m-1)^2\right]}\\ &=(-1)^m\frac{\left(\frac{2m-1}{2}\right)^{2k-2}i^{2k-1}\frac{2k-1}{2}}{(k-m)!(k+m-1)!}. \end{aligned} \tag{24}$$

Therefore, when $n$ is odd, where $n=2k-1, k\in\mathbb{N}_+$,

$$\int_0^\infty\frac{sin^n x}{x^n}dx=\pi i\sum_{m=1}^{k}Res\left[P_n,2(m-1)i\right]= \pi\sum_{m=1}^{k}(-1)^{m+k}\frac{\left(\frac{2m-1}{2}\right)^{2k-2}\frac{2k-1}{2}}{(k-m)!(k+m-1)!}, \tag{25}$$

Let $r=1,3,5,\cdots$ when $n$ is odd, and $r=2,4,6,\cdots$ when $n$ is even. Combining equation (20) and equation (24), the following neat and symmetric equation is obtained:

$$\int_0^\infty\frac{sin^n x}{x^n}dx=\frac{n\pi}{2^n}\sum_r^n(-1)^{\frac{r-n}{2}}\frac{r^{n-1}}{\left(\frac{n-r}{2}\right)!\left(\frac{n+r}{2}\right)!}. \tag{26}$$

## 4. Conclusion

This paper reviewed some classic methods to determine the value of Dirichlet integral, which are Feynman's trick, and residue theorem. Furthermore, this paper extends Dirichlet integral to the case of its $n$-th power, providing an elegant, symmetric solution involving factorials. The solution is based on gamma function, which transforms the extended Dirichlet integral into a double integral, which could be reduced into a rational function via the iteration equation of exponential function times trigonometric function. Then in each case of $n$, the integral of the rational function from zero to positive infinity could be solved by residue theorem, and be combined into one elegant, symmetric equation. Instead of any method related to approximation, this equation could significantly be reduced the amount of time used to determine the value of the extended Dirichlet integral, meanwhile giving a closed form of the solution. In the future, it is hoped that the integral could contribute to calculations of wave functions.

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