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An evaluative overview of the definitions of Cantor set, Fuzzy set and Extension set and their applications, limitation and characteristics

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Abstract:

This article will introduce Cantor sets and their importance in mathematics, especially in areas such as topology and number theory. Also, an overview of how Fuzzy Sets and Extension sets can be useful in areas such as uncertainty handling, decision science, and more. The paper will be organized around the theoretical foundations of Cantor sets, Fuzzy Sets, and Extension sets. The basic principles of construction are presented, as well as important features derived from the construction methods, revealing the advantages as well as the disadvantages that each type of feature exhibits in its application. It will also analyze their performances and connections in different application scenarios. The review will help readers quickly understand an overview of set theory and the fundamentals of the three set theories mentioned above. This paper will introduce the applications of these theories in modern mathematics and computation, engineering, and finance, point out the limitations of the different set theories, respectively, and then provide possible directions for future research.

Keywords: Cantor Set; Fuzzy Set; Intuitionistic Fuzzy Set.

1. Introduction

Set theory is fundamental to Mathematics and can be applied to Computer science, Physics, and Logic because of its characteristics. Russell and Whitehead used the concept of “Set” to define the significance of numbers, then used the First Order Logic to derivate the properties of the quadratic law of operations [1]. Moreover, it serves as a tool for exploring the genuine concept of infinity in greater depth [2]. The set theory defines finite and infinite sets, then establishes the countable infinite and uncountable infinite. This assures the set theory a prominent status in human culture. Because it provides a solid foundation for modern Mathematics, it introduces the concept of “cardinal number and extends the definition in calculus and topology, where the behavior of infinitesimals and infinities is analyzed through limiting

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processes. It also offers elementary works for Gödel's incompleteness theorem and model theory. The traditional set theory (Found by the Cantor set and Zermelo-Fraenkel axiomatic system) has three characteristics: Determinism (the element of a set is definite), Disorder (Elements in a set have no order), and element uniqueness (Each element in a set is unique and will not be repeated). Its accuracy is a double-edged sword. Accuracy leads to exploration in Mathematics but also challenges traditional set theory in solving complex, fuzzy, uncertain problems. So, introducing the none-set concept, Fuzzy set, and Extension set is worth doing. None-set refers to a situation that cannot be determined by binary logic. The size of an object is a great idea for showing fuzziness, but the definition of "huge" may vary from person to person. The classical set theory cannot handle this kind of concept. The fuzzy set can address such complex questions by introducing a membership function that enables the element to belong to the set to a different extent. Membership function given a value between 0 and 1. This allows for the importance of representation, making fuzzy sets suitable for modeling vagueness and uncertainty. The fuzzy set is a core element of extension set theory, which encompasses various forms such as intuitionistic, type-2, and interval-valued fuzzy sets. Cantor sets, fuzzy sets, and extension sets each effectively address different types of problems with their unique characteristics. Those different hallmarks refer to the extension of traditional set theory from various aspects. To begin with, the Cantor set. Cantor set is constructed by removing millions of closed intervals. The Cantor set itself is still an uncountable set; it is infinite. This is an excellent example of how infinite sets can be much larger than countable sets. Cantors Diagonal Argument defines the infinite correctly and gives the size of the infinite. What's more, the Cantor set is an example of a set that is perfect, closed, and nowhere dense, too. This is important for studying the structure of sets and spaces. The existence of the Cantor sets challenges our intuitions about the behavior of infinite sets in continuous spaces. Another property belonging to the Cantor set is Fractal and Self-Similarity. This means every part of it is a copy of itself. It inspired the study of fractals in mathematics, and it also serves as a simple mathematical model of such structures. Then, the Fuzzy set. In a fuzzy set, the membership function assigns a real number between 0 and 1 for each element as mentioned above. The extent of an element belonging to the sets enables the fuzzy set to model real-world situations like tall, happiness, and those human-centered categories. It has higher flexibility and realism than a binary system. The fuzzy set can be widely applied to decision-making systems and artificial intelligence. The last one is the extension set. In this article, we introduce the Intuitionistic Fuzzy Set (IFS), a key concept in extension set theory. First introduced by Atanassov, IFS adds a second parameter, the non-membership degree $v_A(x)$, alongside the membership degree. The sum of these two degrees is constrained to fall between 0 and 1. And the difference between them reflects the degree of uncertainty πA(x). IFS enables more elastin than the Fuzzy Sets by including the uncertainty that the Fuzzy Sets cannot capture. This article will summarize the concepts of Set and Non-Set, Cantor Set, Fuzzy Set, and Extension Set. This paper will reveal their fundamentals, compare their characteristics and relationships, and overview their current applications based on their attributes for subsequent research. To begin with is the definition of Set and Non-Set and their effect. The next step will be to introduce the principle of the Cantor Set, the Fuzzy Set, and the Intuitionistic Fuzzy Set, as well as their features.

2. Organization of the Text

2.1 From Cantor to Fuzzy: Exploring Classical and Modern Set Theories

2.1.1 Sets and non-sets

A set of things is defined as a collection of multiple different objects. Each object in a set is called an element, which is either part of the set or not. No two elements in a set are identical. Set denoted by curly braces {}. A= {c, d, f} means the set named A has three elements: c, d, and f inside. Note that the order of the elements does not matter [3]. This means {c, d, f} and {d, c, f} are the same. The definition of sets refers to five features: 1) clarity: For every element it must be able to be defined inside or outside the set. A= { 1, 2, 3} is firm, so we can say 1, 2, and 3 belong to it. 2) Unorderednessk, as the above. 3) Uniqueness: Each element in a set is unique, meaning no duplicate elements. {1, 2, 2, 3} should be treated as {1, 2, 3}. 4) Membership: An element only has two statues, either belonging to a set or it does not, with no partial membership. 5) Element Identity: Every name must pick out exactly one element. Extensionality is the operating principle of set theory, which guarantees that set theory is self-consistent. Axiom of Extensionality: The two sets A and B are equal if and only if A contains each element of B and B also contains each element of A. It can be expressed more formally like $\forall A \forall B(\forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A=B)$ [4]. Some concepts from classical set theory promote the definition of infinite. The following part will focus on those concepts. Between set and set, they have a different relationship. Subset is one of those relationships; the definition is $S \subseteq S' := \forall x(x \in S \rightarrow x \in S')$. The proper set is an important condition of the subset; it can be defined as S ⊂ S′:= S ⊆ S′ ∧ S≠S’. During the 17th century, Newton and Leibniz discovered calculus. Real numbers are different from the other categories of numbers. They are uncountable and infinite. At that time, calculus used the concept called "infinitesimal." It made the concept of the infinite need to be more precise. Set theory is born to tame the infinite. A set is Dedekind infinite if and only if it is equinumerous with a proper subset of itself. Sets A and B are equinumerous if and only if there is a bijection f: A→B. As shown in the Figure 1.

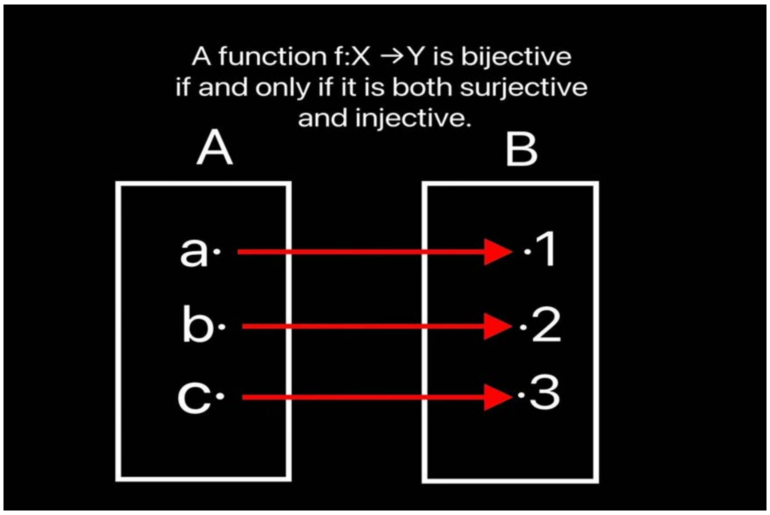

Fig. 1 Schematic diagram of bijection

Non-sets refer to some set theory that can resolve complexity, ambiguity, and uncertainty. Its function leads to consistent features in them. 1) uncertainty: In fuzzy sets, an element might belong to a set at some level. Similarly, intuitionistic fuzzy sets introduce hesitation parameters to represent uncertainty. 2)Hyper-set Properties: the member of the non-set reflects properties of infiniteness or beyond the traditional set limits if they are treated as classical sets, which can cause paradoxes. 3) Dynamism: Non-sets can vary over time. In some probabilistic sets, the probability of elements may change as new information is available. 4) Violation of Classical Set Theory Rules: the non-sets can challenge fundamental principles like the Axiom of Extensionality or the Axiom of Choice. During non-standard analysis, infinitesimal numbers can lie "infinitely close" to real numbers but are not part of the classical actual number set. This article will concentrate on three kinds of non-set theories and introduce their definitions, functions, and features.

2.1.2 Cantor Set, Fuzzy Set, and Extension Set

Let‘s start with the Cantor set. It is made up of points along a line segment, created by repeatedly eliminating the central third from each open interval within the segment. The construction process includes the following sections [5]: 1) Commence with the closed interval [0,1] on the real number line. This interval contains all real numbers between 0 and 1, including both endpoints. 2) Then cut off $I_1^1=(\frac{1}{3},\frac{2}{3})$, the middle third of it. Then leave two lines segments: $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. 3) Next, extract the central third from both of the remaining line segments, both $I_2^1=(\frac{1}{9},\frac{2}{9})$ and $I_2^2=\left(\frac{7}{9},\frac{8}{9}\right)$, get four segments: $\left[0,\frac{1}{9}\right]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$ 4) Repeat this process for each remaining subinterval indefinitely. At the nth time, it has been cut off 2^{n-1} open intervals. The remaining sets are composed of the Cantor set. The construction process of the Cantor Set can be visualized in Figure 2.

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Fig. 2 Cantor Set construction process

The Cantor set has five crucial features that play an important role in its application [6]. 1) Measure Zero. The Lebesgue measure is a way of attributing a value to a subset of the real numbers in a way that generalizes the notion of „length" for intervals. For simple cases, The Lebesgue measure of the interval [a, b] is simply the length of the interval, calculated as b-a. Lebesgue measure, represents the distance, area, or volume of a measurable set, and a set that can be assigned these values is called Lebesgue measurable. The Cantor set is measurable, but its Lebesgue measure is 0. It is easy to understand, for its construction process. Every step will make its length less $\frac{1}{3}$ of itself. Initially, the total length of the interval is 1, after the n times function the length of it will be $\left(\frac{2}{3}\right)^n$. And naturally when n close to ∞, $(\frac{2}{3})^n$ will close to 0. 2)The Cantor set is uncountably infinite, meaning that it contains infinite points, but they cannot be put into bijection relation with the natural numbers set. 3) The Cantor set is nowhere dense, meaning it does not contain any intervals of real numbers. The Cantor set contains no non-trivial open intervals; it comprises isolated points. 4) The canter set is a perfect set. In metric space, a set is perfect if and only if it is a closed set and has no isolated points, in other words, S is perfect if and only if S = S‘ and S‘ is the set consisting of all the limit points of S. 5) The Cantor set is non-empty and has a closed set of boundaries. Take note that during the course of generating the Cantor set, every time specific open intervals are removed, none of the endpoints of the remaining intervals are dug out, and numbers like $\frac{1}{3},\frac{2}{3},\frac{1}{9},\frac{2}{9}$……will ultimately remain in the Cantor set, and hence the Cantor set is a non-empty set. As the complement of the Cantor set is an open set, the Cantor set is therefore considered closed.

$$C^c = (-\infty,0)\cup(1,+\infty)\cup\left(\bigcup_{n=1}^{\infty}\bigcup_{i=n}^{2^n-1} I_n^i\right)$$

Finally, the Cantor set is a non-empty bounded closed set. Next is the Fuzzy Set. A Fuzzy Set is a collection without clear, clearly established edges, permitting components to possess partial inclusion [7]. As we mentioned above, because fuzzy logic allows partial affiliation, the membership function (MF) is a function that assigns every point within the input domain a membership value from 0 to 1. The domain of inputs is also known as the universe of discourse. MF needs to fulfill the only condition that membership values must be between 0 and 1. μA(x) This is called the membership function. It assigns every point of X a degree within the range of 0~1. The MF has various types and these functions are comprised of a few essential functions. 1) Piecewise linear functions 2) Gaussian distribution function 3) Sigmoid curve 4) Quadratic and cubic polynomial curves. The simplicity is the biggest advantage because they are built from straight lines. Then is their logical operation. In standard Boolean logic, its truth tables only have 1 and 0. So, fuzzy logic also conforms to Boolean logic when the Fuzzy values equal 0 or 1. It is the superset of traditional logic.

Boolean	Fuzzy
AND(x,y)	MIN(x,y)
OR(x,y)	MAX(x,y)
NOT(x)	1-x

Fig. 3 A Contrast Between Boolean and Fuzzy Logic Operations

In Fuzzy logic, truth is represented by numerical values, where inputs range from 0 to 1, rather than being strictly true or false. To represent logical operations in fuzzy logic, the AND operation between statements A and B(where A and B are values that fall between 0~1) is handled using the min function, such that A AND B= min (A, B). Similarly, the OR operation can be replaced by the max function, so the A OR B= max (A, B). Last, the NOT operation can be expressed as 1-A. This substitution does not affect the previous truth table. The distinction across the Cantor set and the Fuzzy set operations can be clearly seen in Figure 3. In addition, since there is a membership function, not limited to the truth table alone. Now, this function enables it to take into account values beyond just 0 and 1. The operation of fuzzy logic still requires if-then rules [8,9]. Here is the whole function process: 1) It evaluates every fuzzy statement in the antecedent by assigning a membership value within the range of 0 to 1. 2) If the antecedent consists of a single component, then this supports the rule. If the antecedent consists of multiple parts, fuzzy logic operators are applied to combine them, resulting in one value within [0,1]. This value represents the level of support for the rule. 3) The strength of the whole rule is utilized to define the output fuzzy set. The fuzzy rule's consequent allocates a fuzzy set to the output, described by a membership function that signifies the outcome's accuracy. Fuzzy sets can model continuous variables, providing a more precise representation of the relationship between elements and sets. The fact that the affiliation function can be continuous means that fuzzy sets are flexible enough to handle changing situations in the real world.

The last and the most complex one is the Intuitionistic Fuzzy Sets. This article will not focus on the collection principles but only give an overview. The Intuitionistic Fuzzy Set builds upon and extends the traditional Fuzzy Set described earlier. Apart from the membership degree, it introduces a non-membership degree. Compared with the membership value, $v_A(x)$, the non-membership degree represents the likelihood of an element not belonging to the set. Additionally, the non-membership value must lie between 0 and 1. Combining the two parameters bring about a new value:

Hesitation Degree $\pi_A(x)=1-\mu_A(x)-v_A(x)$.

The explanation demonstrates that it is a value that pictured the uncertainty between membership and non-membership. In other words, this represents the ambiguity or lack of complete information regarding the membership status of an element. Here is the mathematical definition of it, for an intuitionistic fuzzy set A:

$A=\{(x,\mu_A(x),v_A(x))|x\in X)\}$ and $0\leqslant\mu_A(x)+v_A(x)\leqslant 1$

allowing space for hesitation. This approach offers greater flexibility in handling uncertainty, especially when it is unclear whether an element fully belongs to a set.

2.1.3 Comparison and Application

Consider the characteristics of the Cantor set, clarify, simplicity, and definitely, it cannot be used in reality. It is mainly used in mathematical analysis and topology to demonstrate the concepts of infinity, sparsity, and self-similarity. It is often applied to fractal geometry, the study of metric spaces, and the discussion of infinite and uncountability. Although it has fewer direct uses in real-world applications, it plays an essential role in theoretical mathematics, chaos theory, and physics. For example, Cantor sets can be used to construct self-similar fractals. Their self-similar and irregular structures are applied in modeling complex, non-local stochastic processes, partic-

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ularly for processes traditional calculus processes [10].

Then, the Fuzzy Set, the application of μA (x) gives it the ability to deal effectively with fuzziness and uncertainty making it widely used in complex systems. For example, fuzzy sets are often used in decision analysis and control algorithms to cope with multivariate and multicriteria problems such as autonomous driving, artificial intelligence, medical diagnosis, and stock market forecasting [11]. Its features are crucial for environments with incomplete information, especially with handling conflicting data or uncertainty. A current research hotspot combines neural networks and traditional symbolic logic methods called Neuro Symbolic, with some work based on fuzzy logic [12].

Finally, the Institution Fuzzy Set. Introducing non-subordinate and hesitant degrees provides a more powerful tool for further handling uncertainty. This property makes it particularly suitable for applications in image processing in the medical field, financial risk analysis, and multi-criteria decision support systems for complex systems [13].

The Cantor Set generally focuses on infinity and sparsity, mainly used in theoretical mathematics. Fuzzy Set introduces partial membership, allowing it to handle real-world uncertainty, particularly in industrial control and AI. Intuitionistic Fuzzy Set extends fuzzy sets by providing more complex ways to handle uncertainty, making it useful for complex decision-making and risk analysis.

3. Conclusion

Even the Cantor Set promotes Mathematical analysis and fractal geometry. Its self-similarity can construct modeling of physical phenomena. However, its use in practical engineering or applied science is still limited. In future research, combining it with other complex systems (e.g., quantum mechanics, and astrophysics) will be helpful in expanding its application to real-world problems. Although fuzzy sets are superior to traditional logic in dealing with ambiguity, they do not satisfy the need for high precision. Combining it with machine learning applications, such as the introduction of big data, big data models (e.g. Chat-GPT4) may be able to improve their accuracy to solve more complex problems. Introducing two new parameters ensures IFS can deal with complicated issues with enough accuracy. Nevertheless, high accuracy is a two-sided sword. It brings about higher computational complexity and cost. This means it can be too costly to use daily and cannot be applied in real-time decision systems. (e.g., automatic driving) Future research could explore reducing the computational cost of intuitionistic fuzzy sets by optimizing algorithms and computational models. In addition, simplified intuitionistic fuzzy set models can be developed for specific scenarios, retaining the core features but reducing the computational burden.

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