Multipartite Dense Coding in Social Internet of Things

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Abstract

The world today is interconnected. As we live in the contemporary era, we will undoubtedly feel the importance of information interaction in all aspects. This kind of information interaction needs to be carried out or mediated. Quantum communication is a new way of information interaction that has been pushed to the forefront in the past decade. It not only has the low cost of traditional radio communication but also has the characteristics of long relay distance of optical communication system. Compared with these characteristics, the most important thing is that it has super high confidentiality. This is the most critical point in the information age. Dense coding is an important application of quantum information. It reflects the difference between quantum bits and classical bits in transmission, providing a reliable theoretical basis for improving the effectiveness of remote communication in the future. We discuss two kinds of extensions of the original dense coding. They respectively work for high-dimensional and multipartite situations. We give the laws of the two attachments. In the high-dimensional system, we use the Bell state as the target state of any quantum state and realize the one-to-one correspondence between the entangled state and the Bell state. In multipartite space, we can find that the transformation results can form a dimensional space, and the quantum states obtained are one-to-one, corresponding to the classical bit states. In the multipartite system, we can get the transmitted information more efficiently and accurately by limiting data processing by partial senders. At the same time, this restriction will not reduce the amount of information transmitted initially. Therefore, this limited transformation can improve quantum computing power without damaging quantum states.

Keywords: Kronecker product and product state, CPTP map, density matrix, Pauli matrix, dense coding, entangled state, the unitary transformation

ACM Reference Format:
First Author’s Name, Initials, and Last Name, Second Author’s Name, Initials, and Last Name; and Third Author’s Name, Initials, and Last Name. 2018. The Paper’s Title: ACM Conference Proceedings Manuscript Submission Template: This is the paper’s subtitle; this document both explains and embodies the submission format for authors using Word. In Woodstock ‘18: ACM Symposium on Neural Gaze Detection, June 03–05, 2018, Woodstock, NY. ACM, New York, NY, USA, ten pages. NOTE: This block will be automatically generated when manuscripts are processed after acceptance.

1. Introduction

In today’s world, the concept of the interconnection of all things is becoming more and more prominent. Communication between people is more convenient, and the interaction between people and things is more frequent. However, in such an era of high information interaction frequency, ensuring the effectiveness and confidentiality of information transmission has become a significant difficulty in current informatics. In this regard, we can consider the two commonly used communication methods: radio and optical fiber communication. The disadvantages of radio communication are apparent and can be generally summarized as poor confidentiality, short relay distance, small capacity, and weak anti-interference ability. Compared with radio communication, optical fiber has made breakthroughs in these aspects. But the biggest problem is that the cost of optical communication lines is high. The laser source and the optical receiver require a tall fine structure and sensitivity. We are considering whether we can find a communication mode that is different from the two communication modes but can consider both advantages. Quantum communication was born accordingly.

Quantum information is a new subject produced by the intersection of quantum mechanics and informatics. The information carrier of quantum information is a microscopic quantum state, and manipulating the quantum state itself meets the basic principles of quantum mechanics. Therefore, quantum information’s encoding [1], manipulation, transmission, and decoding [2] greatly differ from traditional classical informatics. Based on quantum information technology, we can realize secure quantum communication and solve the problematic computing problems that are difficult to be completed by classical computers [3].

In the terminal 20th century, dense coding was proposed...
by Bennett, a physicist and information theory scholar at IBM, in 1992 [4]. This breaks the original theory that a quantum bit can carry information of at most one classical bit, the so-called Alexander-Holevo boundary [5]. In the following year, to manage the reliable transmission of unknown states, quantum teleportation was proposed [6]. Many quantum communication applications, such as secure quantum key distribution [7][8], have been successfully deployed in recent years. It is reported that photonic qubits are transmitted invisibly over a distance of 1400 kilometers through the uplink channel. But in this article, we introduce one of these applications: dense coding. In the third part of this article (3 Dense codings), we first treat the original dense coding (in a two-bodies two-dimensional system). Then we pointed out that dense coding is inseparable from multiparticle entanglement [9]. The multiparticle entanglement, such as GHZ states [10][11], W states [12], cluster states [13], and genuine multiparticle entangled states [14], are often used in the dense coding as the terminal states after unitary transformation. The terminal conditions can also be written, as Bell states. (not only in two-dimensional space but also the higher space). As for the form of the Bell says, we can refer to the article [15] to gain the Bell states of higher dimensional.

Starting from the mathematical basis of quantum communication, Part 2 Methods, the article introduces the concepts and some properties of Kronecker product, quantum bits, quantum channel (CP map and CPTP map), pure state and mixed state, density matrix, quantum collapse, product state and entangled state, POVM operation, unitary matrix, and unitary transformation and some standard quantum logic gates. The above are the foundations of dense coding.

In Part 3, Dense Coding, we first declare the original dense coding. Then we discuss the dense coding in three-dimensional two-bodies case based on the three-dimensional quantum entangled state (GHZ state) and the three-dimensional Bell states. At the same time, we discuss the three-bodies two-dimensional and the four-bodies dense two-dimensional coding based on the four unitary transformations in the original dense coding to explore the logic of higher dimensional space dense coding.

The rest of the article is Part 4, Conclusion. In this part, we summarize the law in higher bodies and higher-dimensional space’s dense coding and point out that these laws are helpful in obtaining information more quickly and accurately because of the reduction of transformation types.

2. Method

In this section, we introduce the theoretical basics used in this paper. Sec 2.1 reviews the definition of the Kronecker product and its properties. In Sec 2.2, we define the qubit and describe some of its basic properties. In Sec 2.3 and Sec 2.4, we review the definitions of positive semi-definite matrix and positive definite matrix in algebra. Then, we propose the concepts of an entirely positive (CP) map and an utterly positive trace-preserving (CPTP) map. After that, we deduce the situation of the CP map in two body systems (the multipartite system is also applicable); for the CPTP map, we discuss the problem of reversibility and point out that the transformation is mostly irreversible. In Sec 2.5 and Sec 2.6, we introduce the types of common quantum states and a special matrix called the density matrix, used to measure the index of mixed states. We also combine the density matrix with the mixed states and CP map to give another definition of the CP map. In Sec 2.7, we declare the definition of product state and show that the product state and the entangled state are mutually exclusive; this means a state can only be one of the product state or the entangled state. Both can’t hold.

In Sec 2.8, we introduce the positive operator-valued measurement (POVM) operation, one of the general operations, and combine it with quantum collapse to find each state’s corresponding probability of collapse. In Sec 2.9, we give definitions of three logic gates commonly used in quantum calculation. We are sharing their truth table and line expression and discussing their reversibility of them. In Sec 2.10, we give the definitions of unitary transformation. The unitary matrix is a significant basis of dense coding.

2.1 Kronecker product

Definition: Let $A$ be an $n \times p$ matrix and $B$ an $m \times q$ matrix. The $mn \times pq$ matrix

$$A \otimes B = (a_{ij}B)_{mn \times pq}$$

(2.1.1)

It is called the Kronecker product $AB$, and there are some properties of the Kronecker product:

1. Associativity:

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C$$

(2.1.2)

2. Distributivity:

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

(2.1.3)

3. For scalars $\alpha\beta$, we have

$$\alpha A \otimes \beta B = \alpha \beta A \otimes B$$

(2.1.4)

4. For conforming matrices, we can infer

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

(2.1.5)

5. For square nonsingular matrices $AB$, we can define its inverse matrix as:

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

(2.1.9)

2.2 Qubit

A single qubit is a two-state system, such as a two-level
atom. The states $|h\rangle$ and $|v\rangle$ the horizontal and vertical polarization of a photon can also be considered a two-state system.

The underlying Hilbert space for the qubit is $\mathbb{C}^2$. An arbitrary orthonormal basis $\mathbb{C}^2$ is denoted by $\{|0\rangle, |1\rangle\}$. We denoted two orthonormal states of a single qubit as $\{|0\rangle, |1\rangle\}$ where $|0\rangle = |0\rangle = |1\rangle = |0\rangle = 0$, (2.2.1)

Any state of this system can be written as a superposition. $\alpha|0\rangle + \beta|1\rangle$, (2.2.2)

There is a remarkable transformation in this space. It is called the NOT operation (unitary operator), defined as $|0\rangle \rightarrow |1\rangle$, (2.2.3)

2.3 Qubit

In vectors, a square matrix $M$ is positive semi-definite if and only if $\langle a | M | a \rangle \geq 0$ for any vector $a$. But in the language of the map, a CP map refers to a map $K$ which is positive if $K(\rho) \geq 0$ for any $\rho \geq 0$. The CP map includes unit mapping ($K(\rho) = \rho$) and transposition mapping ($K(\rho) = \rho^T$). However, for a general CP map, its form is more complex than unit mapping and transposition mapping. We can take the CP map of a two-body system as an example:

Let $\rho_{AB} = \sum_{j,k,m,n} c_{j,k,m,n} |j,k\rangle\langle m,n|$, (2.3.1)

then

$$(I_A \otimes K_B)\rho_{AB} = (I_A \otimes K_B)(\sum_{j,k,m,n} c_{j,k,m,n} |j,k\rangle\langle m,n|) = (I_A \otimes K_B)(\sum_{j,k,m,n} c_{j,k,m,n} |j\rangle\langle m| \otimes |k\rangle\langle n|) = \sum_{j,k,m,n} c_{j,k,m,n} (I_A \otimes K_B)(|j\rangle\langle m| \otimes |k\rangle\langle n|) = \sum_{j,k,m,n} c_{j,k,m,n} (I_A)(|j\rangle\langle m|) \otimes K_B(|k\rangle\langle n|) = \sum_{j,k,m,n} c_{j,k,m,n} |j\rangle\langle m| \otimes K(|k\rangle\langle n|)$$

Therefore, we can give the CP map a more vivid definition in the two-body system: $K$ is a CP map when $(I_A \otimes K_B)\rho_{AB} \geq 0$ for any $\rho_{AB} \geq 0$. (For the multipartite case, there is a similar derivation.)

2.4 Entirely Positive Trace-Preserving (CPTP) map

CPTP map is proposed based on the definition of CP map. Compared with the CP map, the CPTP map has a special property. That is trace-preserving. The general purpose of a CP map is very similar to the definition of a CP map. The general definition of a CP map is: $K$ is a CPTP map when $K$ is a CP map and $\text{tr}(I \otimes K)\rho_{AB} = \text{tr}(\rho_{AB})$. And most of the time, the CPTP map is an irreversible process, and a few unique maps are invertible operations. It means that:

$\rho \rightarrow K(\rho)$ (2.4.1)

$\rho \rightarrow U \rho U^\dagger$ (2.4.2)

$\rho \rightarrow U \rho U^\dagger \rightarrow \rho$ (invertible operation, a few maps) (2.4.3)

and most of the maps cannot return to $\rho$ (most of the maps) (2.4.4)

2.5 Completely Positive Trace-Preserving (CPTP) map

In quantum mechanics, the state of a microscopic system is determined by the wave function $\psi(r,t)$. A pure state means a single state vector $|\psi\rangle$ in Hilbert space can only describe this system. It means the expression of the form $|\psi\rangle\langle \psi|$. Sometimes, due to statistical physics or quantum mechanics, it is impossible to describe the system with a single state vector. The system is not in a certain state. The only thing we know is that the system may be in a complex state described by $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_n\rangle$ the probability of every $|\psi_j\rangle (j = 1,2,\cdots,n) p_1, p_2, \cdots, p_n (p_j \geq 0)$ is. At this time, we say that the system is in a mixed state, and the system can be expressed as $\sum_j p_j |\psi_j\rangle\langle \psi_j| (\sum_j p_j = 1)$:

As we all know, measuring an observable measurement gets an exact consequence. For the pure state, the observed state $|\psi_j\rangle$ is just the state of its expression, but for the mixed state, the $|\psi_j\rangle, (i = 1,2,\cdots,n)$ is called quantum collapse. Quantum collapse is a phenomenon of mixed states being operated into a single state when observed.

2.6 Density matrix

The density matrix is a concept in quantum statistical
physics. When a quantum system is in a pure state, the state of the system is described by only one state vector; when the system is in a mixed state, the state of the system can be described by the density matrix. The density matrix generalizes the wave function and classical probability distribution. And the density matrix satisfies the unitary evolution equation.

Supposing a mixed state can be described as $\sum_j p_j |\psi_j\rangle\langle\psi_j|$: For any mechanical quantity $A$, the expectation of its measured value $\langle A \rangle$ satisfies

$$\langle A \rangle = \sum_j p_j |\psi_j\rangle \langle A |\psi_j\rangle = \text{Tr}(A \sum_j p_j |\psi_j\rangle\langle\psi_j|),$$

where the $\text{Tr}$ represents tracing the matrix. Thereby defining the density matrix:

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|.$$  

Then we can find that $\langle A \rangle = \text{Tr}(A \rho)$ the density matrix totally determines it.

The pure state can be regarded as a special case of a mixed state whose $i = 1$ (means the state of the system is described by only one $|\psi_i\rangle$) and the corresponding probability $p = 1$.

We can deduce that $|\psi_j\rangle\langle\psi_j|$ this is the density matrix of the pure state $|\psi_j\rangle$.

Here are some density matrix(U) properties in everyday use:

$$U^+ = U^* \text{Tr}(U) = 1$$

The proof of the above three properties is relatively easy. Rather than proving these three theorems, we prefer to propose a typical density matrix here to verify these three properties. The typical matrix is defined as :

$$\Pi_n = \frac{1}{2}(I_2 + \sum_{j=1}^n \sigma_j), \sum_j n_j^2 = 1$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

After simplification,

$$\Pi_n = \frac{1}{2}(1+n_3 \begin{pmatrix} n_1 - n_2i \\ n_1 + n_2i \end{pmatrix} 1-n_3).$$

Therefore,

$$\Pi_n = \left(\prod_{j}^n \sigma_j \right)^* \Pi_n \prod_{j}^n \sigma_j = \frac{1}{2}(1+n_3 \begin{pmatrix} n_1 - n_2i \\ n_1 + n_2i \end{pmatrix} 1-n_3) = \Pi_n$$

$$\text{Tr}(\prod_{j}^n) = 1$$

$$\prod_{j}^n = \frac{1}{4}(1+n_3 n_1 - n_2i 1-n_3)$$

We can see that in addition to the previous two properties, there is a special property here. It is $\prod_{j}^n = \prod_{j}^n$. This property is not only satisfied by this $\prod_{j}^n$ matrix. Supposing the quantum state is comprised of $\sum_j p_j |\psi_j\rangle\langle\psi_j|$. Under the condition of $|\psi_j\rangle = |\psi_k\rangle$ the $\prod_{j}^n$ means that $\sum_j p_j |\psi_j\rangle\langle\psi_j|)^2 = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, expanding the equation, we can obtain the following:

$$\sum_j p_j |\psi_j\rangle\langle\psi_j|)^2 = \sum_j p_j^2 |\psi_j\rangle\langle\psi_j| = \sum_j p_j |\psi_j\rangle\langle\psi_j|.$$  

So we infer that the corresponding coefficients are equal: $p_j^2 = p_j$ and because of $p_j \geq 0$ this, we can deduce that $p_j = 1$, with the $\sum_j p_j = 1$ m, we can assume that $n = 1$ the state of the system can be described as $p_1 |\psi_1\rangle |\psi_1\rangle$ follows. We are sure that this description is the form of the pure state. It means if a state of the system is the pure state, the density matrix(U) of the state has a special property $U^2 = U$.

Combined with the contents of previous Sec 2.3--2.5 (CP(TP) map and mixed state) and the density matrix, we can deduce the following process :

$$U_{\lambda} (\rho \otimes |0\rangle\langle0|_{\lambda}) U^*_{\lambda} = \sigma_{\lambda}$$

$$\Rightarrow \sigma_{\lambda} = \text{Tr}_{\lambda} \sigma_{\lambda}$$

$$= \sum_{i=1}^d (1_\lambda \otimes |i\rangle) \sigma_{\lambda AB}(1_\lambda \otimes |i\rangle)_{\lambda}$$

$$= \sum_j M_j \rho M_j^*$$

In the above formula, the partial trace-over of the system B is defined as $\text{Tr}_{\lambda} \sigma_{\lambda}$. 

So in combination with the pure state and the density matrix’s theory, we can get the Conclusion that a CP map K can always be written as $K(\rho) = \sum_j M_j \rho M_j^*$. 

K
2.7 Product state and entangled state

A system of n-qubits represents a finite-dimensional Hilbert space over complex numbers of dimension $2^n$. A state $|\psi\rangle$ of the system is a superposition of the basic states:

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_n=0}^1 c_{j_1j_2\ldots j_n} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_n\rangle.$$  

(2.7.1)

In a shortcut notation, this state is written as:

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_n=0}^1 c_{j_1j_2\ldots j_n} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_n\rangle.$$  

(2.7.2)

Consider as a particular case the state in the Hilbert space $\mathbb{C}^4$:

$$|\psi\rangle = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle.$$  

(2.7.3)

This state can be written as a product state. The process is as follows:

In Sec 2.1, we can infer that $(A \otimes B)(C \otimes D) = AC \otimes BD$ and the distributivity of Kronecker product. These conclusions can be used to deduce the product state. We can see that:

$$|\psi\rangle = \frac{1}{2} \left[ |00\rangle + |01\rangle + |10\rangle + |11\rangle \right].$$  

(2.7.4)

More generally, for the undetermined coefficient $a, b, c, d$, the state can be expressed as $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ we can use the same way of thinking to deduce when $ad = bc$ this state $|\psi\rangle$ can be described as a product state.

So it is the exact definition of product state: the system’s quantum state can be written as the Kronecker product of its component systems (here, the two-particle quantum system is considered, which can be generalized to the multipartite system). Based on this, we can also define the entangled state. The entangled state is a state of a composite system that may not be written as a product of the states of its component systems.

2.8 Positive Operator-Valued Measurement (POVM) operation

For some applications, the measured state of the system is not of interest, but the main interest lies in the probability of obtaining different measurement results respectively. For instance, the measurement is only performed once before the end of the experiment. Mathematical tools in the form of POVM are particularly suitable for analyzing the measurement results in such cases.

Before we define the form of POVM operation, we can define the completeness relationship first. Consider a state that $|\psi\rangle = \sum_{j=1}^{n} a_j |\psi_j\rangle, \langle \psi_i | \psi_j\rangle = \delta_{ij}, \delta_{ij} = 1(i = j), \delta_{ij} = 0(i \neq j).$ We deduce the corresponding properties of every $|\psi_j\rangle a_j = \langle \psi_i | \psi\rangle i s$. According to the associativity of the Kronecker product, we can infer:

$$\left( \sum_j |\psi_j\rangle \langle \psi_j| \right) |\psi\rangle = \sum_j |\psi_j\rangle \langle \psi_j| = \sum_j a_j |\psi_j\rangle = |\psi\rangle.$$  

From this formula, obviously $\sum_j |\psi_j\rangle \langle \psi_j| = I$. This is the so-called completeness relation, and the $I$ formula is the unit operator.

Based on the completeness relationship, we can define POVM operation as a complete set which is comprised of $\{M_j^+ M_j\}$, and there is a property $\sum_j M_j^+ M_j = I$: The POVM operation means that $\rho \rightarrow \sum_j M_j \rho M_j^+$. Assuming that $\rho$ it is composed of $\rho$, the formula $\rho = \sum_j P_j \rho$, the $P_j y$ can be regarded as the corresponding properties of each state $\rho_j$ when $\rho_j$, orthogonal. We can define the $P_j = Tr(M_j \rho M_j^+)$, and $e \rho$ each can be expressed as:

$$\rho_j = \frac{M_j \rho M_j^+}{Tr(M_j \rho M_j^+)}.$$  

We can assert that $\sum_j P_j = 1$ the certification process is as follows: because of the formula $\rho_j = \frac{M_j \rho M_j^+}{Tr(M_j \rho M_j^+)}$, we can do this transformation: $M_j \rho M_j^+ \rightarrow X$ So the $\rho_j = \frac{M_j \rho M_j^+}{Tr(M_j \rho M_j^+)}$ transform into $\rho_j = \frac{X}{Tr(X)}$. Since the tracing operation is linear, there is $\rho_j = \frac{X}{Tr(X)} \Rightarrow Tr(\rho_j)$.
Because of the commutativity of the tracing operation, we can infer that:
\[
\sum_{j} P_j = \sum_{j} Tr(M_i \rho M_i^*) = \sum_{j} Tr(\rho M_i^* M_i) = Tr(\rho) \sum_{j} M_i^* M_i = Tr(\rho)
\]
Because of \(Tr \rho = 1\), this, the certification \(\sum_{j} P_j = 1\) is accomplished.
Combining with the previous theories in Sec 2.6, we can declare that the POVM operation is a kind of collapse operation, which changes the mixed state into a quantum form in its components. The corresponding quantum state and probability are:
\[
\rho_j = \frac{M_i \rho M_i^*}{Tr(M_i \rho M_i^*)}, \quad P_j = Tr(M_i \rho M_i^*)
\]
For the state of the single-bit system, a common operation is also introduced here. The operation is \(W\) given by:
\[
W[0] = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \\
W[1] = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)
\]
Operation \(W\) is named the Walsh-Hadamard transform. The n-bit Walsh-Hadamard transform is defined as:
\[
W_n := W \otimes W \otimes \cdots \otimes W (n \text{-times})
\]
For example, consider \(n=2\). Find
\[
W_2(|00\rangle) = (W \otimes W)(|0\rangle \otimes |0\rangle) = W|0\rangle \otimes W|0\rangle \\
= \frac{1}{2} (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)
\]
Finally,
\[
W_2(|00\rangle) = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\]
We can see that the Walsh-Hadamard transform generates a linear combination of all states. This also applies to the n-bit system.

**2.9 Controlled-NOT (CNOT) gate, Toffoli(CCNOT) gate and Fredkin gate**

CNOT gate is an important double-bit gate, also known as reversible XOR (exclusive-OR) gate. Its first bit plays a control role and remains unchanged during output. The output of the second bit is an XOR operation. From the properties of the XOR gate, we can infer that:
\[
1 \oplus a = a', \quad 0 \oplus a = a. (\Rightarrow \text{"adverse"})
\]
So when the first bit is 1, the output of the second bit is inverted; when the first bit is 0, the output of the second bit is itself. Because of the efficiency of the first bit, we often call the first bit the control terminal. Therefore we can write the expression of CNOT gate:
\[
a_{\text{output}} = a_{\text{in}}, \\
b_{\text{output}} = a_{\text{in}} \oplus b_{\text{in}}.
\]
There is a one-to-one correspondence between the input and output of the CNOT gate. Therefore, we can infer that the CNOT gate is reversible. Especially the CNOT gate is self-inverse. Because whatever the index of a or b, there is
\[
(a, b) \rightarrow (a, a \oplus b) \rightarrow (a, a \oplus (a \oplus b)) = (a, b).
\]
It means that
\[
(CNOT)^2 = I
\]
In another expression, it means that
\[
(CNOT)^{-1} = CNOT
\]

**Fig 1. Truth table and line representation of the CNOT gate**

The Toffoli gate and the Fredkin gate are based on the CNOT gate. The Toffoli gate is the extension of the CNOT gate in general calculating. It is a three-bit gate. The logic formula is as follows:
\[
a_{\text{output}} = a_{\text{in}}, \\
b_{\text{output}} = b_{\text{in}}, \\
c_{\text{output}} = c_{\text{in}} \oplus (ab)_{\text{in}}.
\]
So we can infer that when \(a=b=1\), the output of c is opposite to itself. The rest are equal to themselves. According to the CNOT gate, we can call the a and b control terminals.

**Fig 2. Truth table and line representation of the Toffoli gate**

The Fredkin gate is also a gate in general calculating. It is called a controllable switching gate because if and only if a is 1, the Fredkin gate exchanges b and c. So we can
get the TT and line expression of Fredkin gate faster than writing the logic formula of Fredkin gate. This also makes the logic formula of Fredkin Gate insignificant.

After seeing the faces and TT of these three gates, we can naturally deduce that the Toffoli and Fredkin gates are also self-inverse like the CNOT gate. It means that

\[(\text{Toffoli})^{-1} = \text{Toffoli}, \quad (\text{Fredkin})^{-1} = \text{Fredkin}.\]

### 2.10 Unitary transformation

If

\[UU^* = U^*U = I\]

Then \(U\) is called unitary. From this definition, we can infer that the adjoint operator of the unitary operator is its inverse, and its adjoint operator is also unitary.

For the product of two unitary operators, it is also unitary because

\[(UV)(UV)^* = U^*VV^*U^* = I\]

### 3 Dense coding

In this part, we discuss a classical problem in quantum communication. That is dense coding. Dense coding is the simplest example of applying quantum entanglement in quantum communication. It allows Alice to send Bob two classical bits of information by sending a single bit. Taking the two-dimensional system as an example, the whole process is as follows:

The first step is that the source generates an EPR pair shared by Alice and Bob. For example, an entangled state: \(|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\) We can transform \(|00\rangle\) to attain the \(|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\). Omitting the specific calculation here, we can get that the transformation is \(\text{CNOT}(H \otimes I)\). This transformation can be written in the form of a matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
1 & 0
\end{bmatrix}
\]

We can conclude that the \(\text{CNOT}(H \otimes I)\) is a unitary transformation.

Further, the sharing of EPR pairs generated by the source in Alice and Bob refers to the source sending one of the EPR pairs to Alice and the other to Bob. There is a significant point in the process: at this time, the distance between Alice and Bob can be ignorable. This means that communication can also be achieved whatever Alice and Bob live in.

Secondly, the two classical bits Alice wants to send to Bob have four possible values: \(00, 01, 10, 11\), which determine the unitary transformation Alice performs on his half of EPR pairs, and the corresponding transformation is \(U = I, \sigma_x, \sigma_y, i\sigma_y\) where the matrix is the Pauli matrix. The expression is: \(\sigma_x = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}, \sigma_y = \begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}, \sigma_z = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\).

The reason for the corresponding transformation is that such a \(U\)-transformation can change the four states of two classical bits into the corresponding four two-dimensional Bell States. The total conditions are as follows:

\(I \otimes I |\phi^+\rangle = |\phi^+\rangle, \quad \sigma_x \otimes I |\phi^+\rangle = |\psi^+\rangle, \quad \sigma_y \otimes I |\phi^+\rangle = |\psi^0\rangle, \quad i\sigma_y \otimes I |\phi^+\rangle = |\psi^-\rangle\).

Therefore, we can obtain the one-to-one correspondence between the unitary transformation, and the Bell states from the above change. This provides a basis for us to transmit the information of two classical bits effectively. Of course, Bob should know the corresponding relationship.

(such as the above: \(|\phi^+\rangle \Rightarrow |00\rangle, |\psi^+\rangle \Rightarrow |01\rangle, |\phi^-\rangle \Rightarrow |10\rangle, |\psi^-\rangle \Rightarrow |11\rangle\)). Between the states of the two classical bits and the Bell states before the operation.

Then Alice passes her half of the EPR pair to Bob. Finally, Bob performs a proper unitary operation on the EPR pair and measures two qubits. First, the obtained Bell state is transformed into the calculated basis vector state. At this time, because of the reversibility of the unitary transformation mentioned in Sec 2.12, we know that the change is the inverse of the change made by the original Alice. According to Sec 2.10, we know that the Hadamard
and CNOT gates are self-reversal. So Bob conducts \((CNOT)(H \otimes I))^{-1} = (H \otimes I)CNOT\) on the Bell states. A matrix can also express this transformation:

\[
B = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0
\end{bmatrix}.
\]

Through this B matrix, we can quickly verify the following:

\[
B|\varphi^+\rangle = |00\rangle,
B|\psi^+\rangle = |01\rangle,
B|\varphi^-\rangle = |10\rangle,
B|\psi^-\rangle = |11\rangle.
\]

Finally, Bob measures the two classical bits on the calculation basis vector to get the two classical bits he wants 100%.

**Fig 3.1 Schematic diagram of dense coding**
(The double line represents two classical bits, and the single line represents one quantum bit)

**Fig 3.2 Quantum circuit representation of dense coding scheme**
For the three-dimensional case, we can also do a similar promotion. Also, consider transitioning from the maximally entangled state to the Bell state. The maximum entangled state (EPR) and the Bell states can be written in the following form:

\[
|\alpha\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) = |\psi_{00}\rangle.
\]

\[
|\psi_{nm}\rangle = \frac{1}{\sqrt{3}} \sum_j e^{\frac{2\pi i n j}{3}} |j\rangle \otimes |j + m \mod 3\rangle
\]

where \(n,m,j = 0,1,2\) and defines \(2+1=0\mod3\) at the same time. We can get nine Bell states:

\[
|\psi_{00}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle),
|\psi_{10}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + e^{\frac{2\pi i}{3}} |11\rangle + e^{\frac{4\pi i}{3}} |22\rangle),
|\psi_{20}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + e^{\frac{4\pi i}{3}} |11\rangle + e^{\frac{2\pi i}{3}} |22\rangle),
|\psi_{01}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |12\rangle + |20\rangle),
|\psi_{11}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + e^{\frac{2\pi i}{3}} |12\rangle + e^{\frac{4\pi i}{3}} |20\rangle),
|\psi_{21}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + e^{\frac{4\pi i}{3}} |12\rangle + e^{\frac{2\pi i}{3}} |20\rangle),
|\psi_{02}\rangle = \frac{1}{\sqrt{3}}(|02\rangle + |10\rangle + |21\rangle),
|\psi_{12}\rangle = \frac{1}{\sqrt{3}}(|02\rangle + e^{\frac{2\pi i}{3}} |10\rangle + e^{\frac{4\pi i}{3}} |21\rangle),
|\psi_{22}\rangle = \frac{1}{\sqrt{3}}(|02\rangle + e^{\frac{4\pi i}{3}} |10\rangle + e^{\frac{2\pi i}{3}} |21\rangle).
\]

So we can figure out the unitary transformation of the EPR to each Bell state. In calculating the base vector, we can infer the matrix \(U\). The unitary matrix is as follows:

\[
U_{nm} \otimes I |\psi_{00}\rangle = U_{nm} \otimes I |\alpha\rangle = |\psi_{nm}\rangle.
\]

\[
U_{00} = I,
U_{10} = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{\frac{2\pi i}{3}} & 0 \\
0 & 0 & e^{\frac{4\pi i}{3}}
\end{bmatrix},
U_{20} = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{\frac{4\pi i}{3}} & 0 \\
0 & 0 & e^{\frac{2\pi i}{3}}
\end{bmatrix},
U_{01} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix},
U_{11} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & e^{\frac{2\pi i}{3}} \\
e^{\frac{4\pi i}{3}} & 0 & 0
\end{bmatrix},
\]
All $U_{nm}$ here is a unitary matrix. Due to the reversibility of the unitary matrix, all operations here can be inverted, which means that Bob can reverse the obtained single quantum bit, which is in the Bell state, into the forms of $\log_2 3^2$ classical bits Alice wants to transmit by a matrix $B$.

This matrix can define a link between the Bell states (in three-dimension) and the conditions of $\log_2 3^2$ classical bits. This operation is also 100% realizable in quantum communication, and the value of the obtained classical bits is 100%.

We can also extend the original dense coding to three-bodies two-dimensional space. It means that there are two people (Alice and Bob) encoding simultaneously and the quantum state received by Charles. But what we need to know is that Charles can only receive the entangled state after Alice and Bob have worked together. How to decode and find the information from Alice and Bob is the biggest problem in this process. In the two-bodies two-dimensional state, we use four typical unitary transformations: $I, \sigma_x, \sigma_z, i\sigma_y$. This can be extended to the three-bodies system. In the three-bodies state, we also try to use these transformations to encode the particles of Alice and Bob. There are 16 combinations of Alice and Bob’s change. At the same time, dense coding must find the maximum entangled state in the three-bodies system. According to the two-bodies system, we can show that $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is the largest entangled state in the three-bodies system. Then we can express the total change as follows:

$$ (U_i \otimes U_j \otimes I)|GHZ\rangle = |\psi_i \rangle, i = 1, 2, 3, 4, j = 1, 2, \ldots, 16 $$

And the $U_i$ means $U_1 = I, U_2 = \sigma_x, U_3 = i\sigma_y, U = \sigma_z$.

Based on this basis, we can infer:

$$ (I \otimes I \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = |GHZ\rangle; $$

$$ (I \otimes \sigma_x \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle), $$

$$ (I \otimes i\sigma_y \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|011\rangle - |100\rangle), $$

$$ (I \otimes \sigma_z \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle), $$

$$ (\sigma_x \otimes I \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|110\rangle + |001\rangle), $$

$$ (\sigma_z \otimes I \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle), $$

$$ (\sigma_z \otimes i\sigma_y \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |101\rangle), $$

$$ (\sigma_z \otimes \sigma_z \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |110\rangle), $$

$$ (i\sigma_y \otimes I \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|110\rangle - |001\rangle), $$

$$ (i\sigma_y \otimes \sigma_y \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |101\rangle), $$

$$ (i\sigma_y \otimes i\sigma_y \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle), $$

$$ (\sigma_z \otimes \sigma_z \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|110\rangle + |001\rangle), $$

$$ (\sigma_z \otimes \sigma_z \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle), $$

$$ (\sigma_z \otimes \sigma_z \otimes I)|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). $$

The sixteen states obtained here can also be classified. Considering the constant uncertainty and phase uncertainty of the wave function, we can change the above sixteen states into eight states:

$$ |\psi_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), $$

$$ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle), $$

$$ |\psi_3\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle), $$

$$ |\psi_4\rangle = \frac{1}{\sqrt{2}}(|100\rangle - |011\rangle). $$
This means that the sixteen changes actually generate an eight-dimensional space. Each two of these eight quantum states are orthogonal to one, which can form an eight-dimensional area. This means the 16 changes may not need to be fully edited. Compared with the original dense coding, the biggest problem of the three-body dense coding is getting the information that Alice and Bob want to convey simultaneously. This means that if we can reduce the changes made by Alice or Bob, we can more effectively judge the information that Alice and Bob originally wanted to convey. Combining the above eight quantum states and sixteen unitary changes, we find some strange laws: as long as one of Alice and Bob takes all four changes, the other only needs to take two of the four changes as the processing method. Such a joint operation can take all the expressed quantum states and correspond to the eight-dimensional space formed by these quantum states one-to-one. This means that this space, which a set of eight basis vectors can represent, can be decomposed into a Kronecker product of an area that can be represented by two basis vectors and an area that can be represented by for basis vectors. Since the whole process transformation is unitary, due to the reversibility of the unitary transformation, we know that the finally obtained qubit can be changed back to the original classical bit state through a reversible matrix. Suppose Charles knows which two encoding methods Alice uses. In that case, it can infer the classical bit information transmitted by Alice and Bob to Charles based on the received quantum bit state, realizing the one-to-one correspondence between the quantum bit state and the classical bits.

\[
|\psi_s\rangle = \frac{1}{\sqrt{2}} (|010\rangle + |101\rangle), \\
|\psi_a\rangle = \frac{1}{\sqrt{2}} (|010\rangle - |101\rangle), \\
|\psi_r\rangle = \frac{1}{\sqrt{2}} (|001\rangle + |110\rangle), \\
|\psi_l\rangle = \frac{1}{\sqrt{2}} (|001\rangle - |110\rangle).
\]

Since the three-body system can do this, can the four-body system do the same? Here we can further expand the system’s volume to a four-bodies system. According to the Conclusion of the original dense coding and the three-bodies dense coding, the space formed by the quantum states obtained by the four-bodies dense coding here is $2^4 = 16$ dimensional. We know that the maximum entangled state in the four bodies is
\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle).
\]

If we take the same four unitary transformations, there should be 64 quantum states (not combined, not subject to constant factor or phase change) here. We can find out all these transformations:

<table>
<thead>
<tr>
<th>State</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>0000\rangle$</td>
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<td>$</td>
<td>0100\rangle$</td>
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<td>1110\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>1111\rangle$</td>
</tr>
</tbody>
</table>

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle).
\]

Fig 3.3 three-bodies situation of dense coding
(the S means source, A means Alice, B means Bob, C means Charles)
\[
(\sigma_x \otimes \iota \otimes \iota \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0101\rangle - |1010\rangle \right), (\sigma_x \otimes \iota \otimes \iota \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |1000\rangle - |0111\rangle \right),
\]
\[
(\sigma_x \otimes \sigma_x \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0011\rangle + |1100\rangle \right), (\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle + |1110\rangle \right),
\]
\[
(\sigma_x \otimes \sigma_x \otimes \iota \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle - |1110\rangle \right), (\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |1100\rangle - |0011\rangle \right),
\]
\[
(\sigma_x \otimes \iota \sigma_x \otimes \iota \otimes \iota) |\varphi\rangle = -\frac{1}{\sqrt{2}} \left( |1100\rangle - |0011\rangle \right), (\sigma_x \otimes \iota \sigma_x \otimes \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle - |1110\rangle \right),
\]
\[
(\sigma_x \otimes \iota \sigma_x \otimes \iota \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle + |1110\rangle \right), (\sigma_x \otimes \iota \sigma_x \otimes \sigma_x \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0101\rangle - |1010\rangle \right),
\]
\[
(\sigma_x \otimes \sigma_z \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0101\rangle + |1010\rangle \right), (\sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0111\rangle + |1000\rangle \right),
\]
\[
(i \sigma_y \otimes \iota \otimes \iota \sigma_y \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0111\rangle - |1000\rangle \right), (i \sigma_y \otimes \iota \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0011\rangle + |1000\rangle \right),
\]
\[
(i \sigma_y \otimes \sigma_y \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0111\rangle - |1100\rangle \right), (i \sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle - |1110\rangle \right),
\]
\[
(i \sigma_y \otimes i \sigma_y \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |1001\rangle + |0110\rangle \right), (i \sigma_y \otimes i \sigma_y \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle + |1110\rangle \right),
\]
\[
(i \sigma_y \otimes i \sigma_y \otimes i \sigma_y \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0111\rangle - |1100\rangle \right), (i \sigma_y \otimes i \sigma_y \otimes i \sigma_y \otimes \sigma_z) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |1100\rangle + |0011\rangle \right),
\]
\[
(i \sigma_y \otimes i \sigma_z \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle + |1110\rangle \right), (i \sigma_y \otimes i \sigma_z \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0101\rangle - |1010\rangle \right),
\]
\[
(i \sigma_y \otimes i \sigma_z \otimes i \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0111\rangle - |1000\rangle \right), (i \sigma_y \otimes i \sigma_z \otimes i \sigma_z \otimes \sigma_z) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0011\rangle - |1000\rangle \right),
\]
\[
(\sigma_z \otimes \sigma_z \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0001\rangle - |1111\rangle \right), (\sigma_z \otimes \iota \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0010\rangle - |1101\rangle \right),
\]
\[
(\sigma_z \otimes \iota \otimes \iota \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0010\rangle + |1101\rangle \right), (\sigma_z \otimes \iota \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0000\rangle + |1111\rangle \right),
\]
\[
(\sigma_z \otimes \sigma_z \otimes \iota \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0100\rangle - |1011\rangle \right), (\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0110\rangle - |1001\rangle \right),
\]
\[
(\sigma_z \otimes \sigma_z \otimes \iota \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0110\rangle + |1001\rangle \right), (\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \iota) |\varphi\rangle = \frac{1}{\sqrt{2}} \left( |0011\rangle - |1000\rangle \right),
\]
Based on the above 64 changes, we can get a similar conclusion as the three-bodies system: the receiver can finally get sixteen orthogonal normalized quantum states through constant change and phase change. Four joint transformations obtain each of the sixteen quantum states. If Alice, Bob, and Charles adopt the same probability of each shift, the appearance of the sixteen quantum states will also be an event of equal chance. These sixteen quantum states form a 16-dimensional space. Because of the four unitary transformations of the original dense coding, the 64 changes can be further simplified here. It can be turned into the 16 changes: Alice only needs two, Bob only needs two, and Charles takes all four shifts. There is no limit to the two changes Alice and Bob can make here, whichever are the same. At the same time, Alice’s changes do not affect Bob’s changes. In other words, Alice’s changes have nothing to do with Bob’s changes. For example, if Alice takes the first two changes, Bob can also take the first two instead of only taking the last two shifts. Suppose the receiver knows in advance what the two unitary transformations of Alice and Bob are. In that case, he can judge what information Alice, Bob, and Charles are sending by the received quantum state. Obviously, the correspondence of this judgment is also one-to-one, and there will be no case where one received quantum state can correspond to two classical bits. We know that the whole transformation process is unitary. Since unitary transformation can be realized in quantum operation and has reflexivity, we assert that we can find a matrix $R$ similar to matrix $B$ in the original dense coding so that the quantum state obtained by the receiver can be transformed back to the state of classical bits.

$$\begin{align*}
A & \rightarrow S \\
B & \rightarrow C \\
C & \rightarrow D
\end{align*}$$

Fig 3.4 Four-bodies situation of dense coding
(the S, A, B, and C are the same as the three, but the difference is at this time, Charles is the operator, and the D means receiver)

4 Conclusion
In the article, we have shown the situation of original dense coding and two extensions of dense coding: higher-dimensional situation and multipartite situation. We offer the change from the initial state to the Bell state in the case of a high-dimensional system and point out that the matrix from the Bell state to the classical bit state is realizable. After that, we considered that the senders may not be single. There might be multiple senders transmitting messages to the same destination. So we thought of the dense coding process of three-bodies or even four-bodies systems. We discovered the law of quantum bit changes in three-bodies and four-bodies systems, which could effectively reduce the transformations. It also solved the problem that the quantum state obtained by the receiver in the original three-body and four-body cannot uniquely determine the information sent by the original source (for instance, in the four-bodies system, if we did not cut down the forms of transformation of the unitary matrix, we known every terminal states can be corresponding to four kinds of unitary, this meant that the classical bit was unable to determine). This inspired the current field of quantum communication and quantum computing. This restriction on transformation ensured the integrity of information and simplified the operation and detection process. If it can be extended to more complex multipartite systems, it will undoubtedly improve the computing power of quantum computers and make quantum communication channels more simple.

However, this article also has a lot of content worth further mining. In the higher-dimensional dense coding, we found the law of unitary transformation, but we did not express matrix $B$, which can turn the quantum bits into classical bits. We had just testified the situation of three-dimensional space, not verifying our prediction again in a higher-dimensional area. The higher-dimensional room may have the same law: every classical state can be changed into a Bell state by a series of unitary transformations and finally return to the original classical bit. In the multipartite space, we also just thought about the circumstance of three-bodies or four-bodies distance; as for more bodies space, we still need to give proof of it. After studying the three-bodies and four-bodies systems, we found that the dimension of the final quantum state was $2^n$ (n means how many bodies). The size of this space can correspond to the simplified transformation one by one. But this is based on the fact that the receiver knows the type of unitary transformation made by Alice et al. If Alice and others do not cooperate with the receiver and do not tell the receiver what kind of transformation they are making, this result may not be applicable. Based on the research of this article, this dilemma of higher-dimensional or many-body is hopeful to work out. But the situation still needs to be discovered by the receiver and still needs interested and talented scholars or experts to research it.

We expect more scholars or researchers to continue to engage in research on dense coding.

REFERENCES