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Comparative Analysis of Portfolio Optimization under Various Constraints

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Abstract:

The pursuit of optimal portfolio construction, whether through maximizing expected returns under a fixed level of risk or minimizing risk for a given expected return, constitutes a fundamental aspect of portfolio management. In this study, we analyzed essential statistical metrics for 22 risky assets including 1 index stock, checked about several assumptions, and applied portfolio optimization methods under different regulation requirements to compare the models between Markowitz Model (MM) and the Index Model (IM). We identified key points on the efficient frontier, the Capital Allocation Line (CAL), and feasible portfolios, and we came to the conclusion that MM model is more favorable compared to IM model in this case. By reviewing the different settings of each problem, a sensitivity analysis was also carried out to analyze the effect of regulation on the weights of our portfolio, risk and return, which is critical in practice for investors. This research not only contributes to the empirical literature on the MM and IM models but also offers practical investment recommendations for portfolio construction.

Keywords: Markowitz model, Index model, portfolio optimization, regularization

1. INTRODUCTION

The 21st Century has witnessed a booming in capital markets, yet the rapid changes in geopolitics and volatility in assets have brought the problem of protecting personal wealth to the public. Citizens and entrepreneurs all followed Nvidia and OpenAI for the past few years, and wealth has been redistributed from traditional industries to high-tech industries. Such transformation is driven by the desire to preserve or grow wealth in an inflationary environment, and the update of modern financial theories that have been evolving with markets. Nowadays investors are equipped with more tools, and more options to choose. Mutual fund, closed fund, hedge fund and various funds provide different risk preference for different investors, and the innovation in finance like Bitcoin, ETH or encouraged investors to take part in the trading process. With such diversified options for asset allocation, investors face more risk and opportunities, which therefore highlight the importance of risk management and the need for robust portfolio optimization strategies to mitigate losses and enhance returns.

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The principles of portfolio risk management is to minimize the risk while maximizing the return. Since Markowitz Model first introduced in 1952, the strategies have been evolving. Markowitz provided a framework to construct portfolios with minimal variance for certain level of expected return, and he proved one important fact: Don't put all your eggs in one basket. By recognizing risk as systematic risk, which arises from macroeconomic factors and cannot be eliminated, and unsystematic risk, which can be diversified away through asset allocation, the risk can be reduced to systematic risk which matches investors' risk tolerance.

In 1963, William Sharpe addressed these limitations by proposing the Single Index Model (IM), which simplifies the estimation of covariance matrices and enhances the prediction of security risk premiums. The IM reduces the computational complexity of the MM by relating asset returns to a single market index, making it more practical for real-world applications.

For now, with Fama-French 3 factors, and later Carhart 4 factors to Fama-French 4 factors, factor models are more popular over the last two decades. Researchers have been looking for more factors, and hedge fund managers have been seeking alpha to beat the markets. With modern factor theories, portfolio management has also embraced the development of new technology like Deep Learning, Barra factor models, and all kinds of modern frameworks.

In this study, however, we focus mainly on the ground of portfolio management, that is to say, we empirically examines the performance of the MM and IM under various regulatory and optimization constraints, with a particular focus on the inclusion or exclusion of the S&P 500 as a broad equity index. By using historical data from 21 well-known stocks, we construct portfolios under both models and analyze the feasible portfolio regions, including the efficient frontier, the minimal risk frontier, and the minimal return frontier. Here we not only analyzed the normality assumption and solved the problems under different constraints, but also work on the regulation government imposed, and the effect brought to the investors.

Our research contributes to the empirical literature on portfolio optimization by comparing the MM and IM under different constraints and providing practical insights for investors. By leveraging historical data, we offer some insights for constructing portfolios that align with investors' risk preferences and regulatory requirements, ultimately helping to reduce investment risks and enhance returns. Our results also cast light on the regulations, which may also help governments make decisions, and for FoF investors to reconsider the potential risk they may face when regulations are revised.

2. THEORY

2.1 The Markowitz Model

The Markowitz Mean-Variance Model (MM), introduced by Harry Markowitz in 1952, is a foundational framework in modern portfolio theory. It provides a mathematical approach to constructing portfolios that optimize the trade-off between risk and return. The model is based on the following key assumptions:

1. Investor Behavior: Investors evaluate investment opportunities based on the probability distribution of security returns over a given holding period. This implies that investors are rational and make decisions based on expected outcomes.
2. Risk Measurement: Investors assess the risk of a portfolio using the variance or standard deviation of expected returns. Variance measures the dispersion of returns around the mean, providing a quantitative measure of risk.
3. Risk-Return Trade-off: Investment decisions are made solely on the basis of risk and return. Investors aim to maximize expected returns for a given level of risk or minimize risk for a target level of expected return.
4. Optimization Objective: At a given level of risk, investors seek to maximize expected returns. Conversely, for a given level of expected return, investors aim to minimize portfolio risk.

The expected return of a portfolio P in the Markowitz model is calculated as the weighted average of the expected returns of the individual securities:

$$E\left(R_P\right)=\sum_{i=1}^{n} w_i E\left(R_i\right)$$

Where $E\left(R_P\right)$ represents Expected return of the portfolio, w_i represents the Proportion of asset i in the portfolio, and $E\left(R_i\right)$ Expected return of asset i.

The standard deviation (risk) of the portfolio is given by:

$$\sigma_P=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(R_i,R_j)}=\sqrt{\vec{w}\Sigma\vec{w}}$$

Where σ_P represents Standard deviation of the portfolio, and $Cov(R_i,R_j)$ the Covariance between the returns of assets i and j. In matrix format, the $\vec{w}$ represents the weight vector of portfolio, and Σ the covariance matrix of assets. The covariance matrix captures the degree to which asset returns move together, reflecting diversification benefits. By minimizing portfolio variance for a given level of expected return, the Markowitz model identifies the efficient frontier, which represents the set of optimal portfolios offering the highest expected return for a given level of risk.

2.2 The Single Index Model

The Single Index Model (IM), developed by William Sharpe in 1963, simplifies the Markowitz model by reducing the complexity of estimating covariance matrices. The IM is based on two key assumptions:

1. Systematic and Idiosyncratic Risk: The risk of a security is divided into systematic risk (market risk) and idiosyncratic risk (firm-specific risk). Systematic risk is driven by macroeconomic factors (e.g., market index movements), while idiosyncratic risk is specific to individual securities and can be diversified away.
2. Independence of Idiosyncratic Risks: The idiosyncratic risk of one security is uncorrelated with that of another security. This implies that the returns of two securities are correlated only through their joint response to systematic factors like market index (SPX).

These assumptions significantly simplify the calculation of portfolio risk, as the covariance between securities is determined solely by their relationship with the market index.

The expected return of a portfolio P in the IM is the same with MM given by:

$$E(R_P) = \sum_{i=1}^{n} w_i E(R_i)$$

The standard deviation of the portfolio is given by:

$$\sigma_P = \sqrt{\sum_{i=1}^{n} w_i^2 \beta_i^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{\epsilon_i}^2}$$

Where β_i represents Beta of asset i, measuring its sensitivity to the market index, σ_M represents Standard deviation of the market index, and σ_{ϵ_i} Standard deviation of the idiosyncratic risk of asset i.

The IM simplifies the covariance matrix by expressing security returns as a linear function of the market index:

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i$$

Where α_i represents Intercept term (asset-specific return not explained by the market), and ϵ_i Idiosyncratic error term.

By regressing historical security returns $R_i(t)$ onto the market index returns $R_M(t)$, we obtain the regression equation:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + \epsilon_i$$

This regression allows us to estimate the systematic and idiosyncratic components of risk, facilitating portfolio optimization.

2.3 Comparison of MM and IM

To compare the performance of the Markowitz Model (MM) and the Single Index Model (IM), We construct a complete portfolio C by combining risky assets with a risk-free asset. Let y represents the proportion of the portfolio allocated to risky assets and 1-y the proportion allocated to the risk-free asset. The expected return of the complete portfolio is:

$$E(R_C) = yE(R_P) + (1-y)R_f$$

Where R_f represents Risk-free rate of return.

The standard deviation of the complete portfolio is:

$$\sigma_C = y\sigma_P$$

The Capital Allocation Line (CAL) represents the set of portfolios combining risky and risk-free assets. The slope of the CAL, known as the Sharpe Ratio, measures the excess return per unit of risk:

$$SharpeRatio = \frac{E(R_p) - R_f}{\sigma_P}$$

The Minimum-Variance Frontier is the set of portfolios with the lowest variance for a given level of expected return. The Efficient Frontier consists of portfolios on the minimum-variance frontier that offer the highest expected return for a given level of risk. The Global Minimum-Variance Portfolio is the portfolio with the lowest possible risk, while the Optimal Risky Portfolio is the point where the CAL is tangent to the efficient frontier, offering the highest Sharpe Ratio.

While the optimization problem is solved by Excel Solver, it's actually an convex optimization problem which means we can solve it with gradient descending method, which is easy for non-linear problems.

3. EXPLORATORY DATA ANALYSIS & PRE-PROCESS

3.1 Data Description

To empirically validate the theoretical foundations of portfolio optimization models, we selected a sample of 21 stocks spanning four distinct equity sectors, as classified by Yahoo Finance: Technology, Financial Services, Communications, and Consumer Finance. These sectors were chosen to ensure diversification and to capture the varying risk-return profiles associated with different industries. Additionally, we included the S&P 500 Index as a broad market benchmark and the 1-month Federal Funds Rate as a proxy for the risk-free rate, resulting in a total of 22 risky assets for our analysis.

The historical daily price data for these stocks and the

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S&P 500 Index were obtained from Bloomberg Professional, covering a 20-year period from September 17, 2004, to September 20, 2024. To ensure consistency and eliminate non-trading days, we filtered the data to include only five working days per week. The selected stocks, along with their respective sectors and ticker symbols, are presented in Table 1. This diversified sample allows us to examine the impact of sector-specific risks and correlations on portfolio construction.

Table 1. 21 stocks fundamentals and prices

Stock Code	Stock Name	Industry	Price Movement	P/E
AMZN	AMAZON.COM INC	Communications		44.97
BKNG	BOOKING HOLDINGS INC	Communications		25.27
GOOGL	ALPHABET INC-CL A	Communications		23.1
NFLX	NETFLIX INC	Communications		45.68
MCD	MCDONALD'S CORP	Consumer, Cyclical		25.56
SBUX	STARBUCKS CORP	Consumer, Cyclical		26.14
TGT	TARGET CORP	Consumer, Cyclical		15.58
WMT	WALMART INC	Consumer, Cyclical		33.91
BAC	BANK OF AMERICA CORP	Financial		13.95
BK	BANK OF NEW YORK MELLON CORP	Financial		13.65
C	CITIGROUP INC	Financial		17.53
GS	GOLDMAN SACHS GROUP INC	Financial		15.82
AAPL	APPLE INC	Technology		34.73
ACN	ACCENTURE PLC-CL A	Technology		28.34
IBM	INTL BUSINESS MACHINES CORP	Technology		23.61
AMD	ADVANCED MICRO DEVICES	Technology		175.93
INTC	INTEL CORP	Technology		44.65
INTU	INTUIT INC	Technology		57.97
NVDA	NVIDIA CORP	Technology		56.61
COF	CAPITAL ONE FINANCIAL CORP	Financial		13.4
SCHW	SCHWAB(CHARLES) CORP	Financial		23.92

It is evident that the technology sector typically exhibits a high Price-to-Earnings (P/E) ratio, which may indicate that the stocks in this sector are either overvalued or reflect high growth expectations from investors. A high P/E ratio often suggests that investors are willing to pay a premium for future earnings potential, driven by innovation, rapid growth, and disruptive technologies. In contrast, the financial sector tends to have a relatively low P/E ratio, reflecting more conservative growth expectations and a focus on stability and dividend yields rather than aggressive expansion.

Analyzing the price movements since 2004, we observe distinct trends across sectors. For the technology, internet, and consumer sectors, stock values have shown a consistent upward trajectory, driven by technological advancements, increasing digitalization, and strong consumer demand. These sectors have benefited from global trends such as the rise of e-commerce, cloud computing, and the proliferation of digital services, which have fueled investor optimism and driven valuations higher.

On the other hand, the financial sector has exhibited a more gradual slope in its price movements. This slower growth can be attributed to the sector's sensitivity to macroeconomic factors such as interest rates, regulatory changes, and economic cycles. Financial institutions, including banks and insurance companies, often face tighter regulatory scrutiny and are more exposed to systemic risks, which can limit their growth potential compared to high-growth sectors like technology. Additionally, the financial sector's performance is closely tied to the broader economy, making it less volatile but also less likely to experience the explosive growth seen in technology-driven industries.

These contrasting trends highlight the importance of sector diversification in portfolio construction. While high-growth sectors like technology offer the potential for significant returns, they also come with higher volatility and valuation risks. In contrast, the financial sector provides more stable returns but may lack the same growth momentum.

3.2 Exploratory Data Analysis

We use Python 3.8 to calculate and exhibit data of each stocks. The data is aggregated into monthly and weekly frequency, as shown in Table 2.

Table 2.1. Daily stock data statistics

	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	BAC	BK
count	5220	5220	5220	5220	5220	5220	5220	5220	5220	5220	5220
mean	0.05%	0.11%	0.13%	0.09%	0.16%	0.06%	0.06%	0.05%	0.04%	0.05%	0.05%
std	1.19%	2.36%	2.38%	1.89%	3.19%	1.24%	1.91%	1.89%	1.24%	2.87%	2.19%
min	-11.98%	-21.85%	-17.28%	-11.63%	-40.92%	-15.88%	-16.2%	-24.93%	-11.38%	-28.97%	-27.16%
25%	-0.38%	-0.94%	-0.93%	-0.75%	-1.26%	-0.52%	-0.8%	-0.77%	-0.53%	-0.92%	-0.78%
50%	0.05%	0.0%	0.02%	0.02%	0.0%	0.04%	0.0%	0.0%	0.01%	0.0%	0.0%
75%	0.56%	1.18%	1.12%	0.95%	1.51%	0.65%	0.89%	0.86%	0.62%	0.97%	0.9%
max	11.58%	26.94%	26.15%	19.99%	42.23%	18.13%	24.5%	20.43%	11.71%	35.27%	24.81%
SKEW	-0.26	0.87	1.1	0.63	-0.11	0.38	0.69	0.13	0.24	0.93	0.62
KURT	13.37	15.54	14.81	9.76	23.6	19.78	13.94	17.34	13.57	28.32	22.28

	C	GS	AAPL	ACN	IBM	AMD	INTC	NVDA	COF	SCHW	INTU
count	5220	5220	5220	5220	5220	5220	5220	5220	5220	5220	5220
mean	0.02%	0.06%	0.14%	0.07%	0.04%	0.11%	0.03%	0.18%	0.06%	0.07%	0.08%
std	3.03%	2.15%	2.01%	1.6%	1.4%	3.55%	2.01%	3.02%	2.72%	2.28%	1.84%
min	-39.02%	-18.96%	-17.93%	-13.45%	-12.85%	-26.18%	-26.06%	-30.72%	-25.04%	-15.17%	-14.48%
25%	-0.93%	-0.86%	-0.78%	-0.66%	-0.59%	-1.6%	-0.89%	-1.29%	-0.97%	-1.01%	-0.73%
50%	0.0%	0.0%	0.05%	0.04%	0.0%	0.0%	0.0%	0.06%	0.0%	0.0%	0.05%
75%	0.92%	1.01%	1.13%	0.83%	0.71%	1.81%	0.98%	1.6%	1.04%	1.15%	0.93%
max	57.82%	26.47%	13.91%	16.37%	11.52%	52.29%	19.52%	29.8%	26.43%	19.65%	20.08%
SKEW	1.51	0.79	0.07	0.16	-0.23	0.68	-0.55	0.23	0.31	0.32	0.23
KURT	51.74	19.11	5.81	8.44	9.17	13.16	13.27	8.08	16.36	7.42	9.06

Table 2.2. Weekly stock data statistics

	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	BAC	BK
count	1043	1043	1043	1043	1043	1043	1043	1043	1043	1043	1043
mean	0.22%	0.56%	0.65%	0.47%	0.82%	0.32%	0.32%	0.24%	0.22%	0.24%	0.21%
std	2.53%	5.27%	5.25%	4.28%	7.5%	2.59%	4.26%	4.05%	2.61%	6.57%	4.3%
min	-13.83%	-21.62%	-19.86%	-14.22%	-43.37%	-20.26%	-18.6%	-29.51%	-17.28%	-37.18%	-19.85%
25%	-0.84%	-2.11%	-2.27%	-1.86%	-2.89%	-1.04%	-1.79%	-1.84%	-1.19%	-2.14%	-1.68%
50%	0.35%	0.56%	0.71%	0.57%	0.61%	0.35%	0.17%	0.28%	0.25%	0.19%	0.22%
75%	1.46%	3.29%	3.17%	2.68%	4.03%	1.7%	2.28%	2.13%	1.65%	2.45%	2.23%
max	17.43%	36.98%	32.05%	25.64%	63.47%	22.63%	30.66%	29.72%	12.68%	64.8%	25.75%
SKEW	-0.14	0.54	0.57	0.35	0.65	0.03	0.77	0.44	-0.33	1.33	0.34
KURT	6.66	5.28	4.53	2.85	9.16	9.59	7.62	8.07	4.11	18.25	5.95

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	C	GS	AAPL	ACN	IBM	AMD	INTC	NVDA	COF	SCHW	INTU
count	1043	1043	1043	1043	1043	1043	1043	1043	1043	1043	1043
mean	0.09%	0.29%	0.68%	0.34%	0.2%	0.57%	0.15%	0.9%	0.27%	0.34%	0.41%
std	7.34%	4.59%	4.6%	3.42%	3.16%	8.14%	4.36%	6.81%	6.03%	4.99%	3.81%
min	-43.93%	-23.31%	-19.68%	-13.56%	-15.9%	-32.68%	-34.77%	-35.36%	-30.35%	-32.23%	-20.19%
25%	-2.35%	-2.03%	-1.94%	-1.6%	-1.3%	-3.83%	-1.98%	-2.81%	-2.16%	-2.38%	-1.61%
50%	0.23%	0.39%	0.63%	0.49%	0.23%	0.22%	0.28%	0.8%	0.33%	0.4%	0.31%
75%	2.4%	2.65%	3.28%	2.24%	1.86%	5.14%	2.61%	4.87%	2.62%	2.92%	2.52%
max	121.9%	26.98%	23.8%	18.19%	19.16%	35.8%	17.03%	30.69%	49.84%	24.76%	22.25%
SKEW	4.18	0.07	0.05	-0.12	-0.13	0.2	-0.7	-0.02	0.89	-0.21	-0.01
KURT	77.95	3.93	1.85	2.65	4.14	1.84	5.22	2.33	10.9	4.95	3.11

Table 2.3. Monthly stock data statistics

	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	BAC	BK
count	241	241	241	241	241	241	241	241	241	241	241
mean	0.93%	2.4%	2.78%	2.03%	3.49%	1.33%	1.3%	1.0%	0.92%	0.84%	0.77%
std	4.31%	10.4%	10.62%	8.59%	15.12%	4.65%	7.8%	7.9%	4.94%	11.7%	6.87%
min	-16.8%	-30.46%	-27.05%	-18.48%	-51.8%	-14.84%	-31.99%	-28.91%	-15.95%	-53.27%	-17.68%
25%	-1.58%	-4.51%	-3.88%	-3.49%	-5.52%	-1.32%	-3.47%	-3.69%	-2.29%	-4.29%	-3.55%
50%	1.44%	2.3%	2.09%	1.87%	3.12%	1.29%	1.49%	0.44%	1.07%	0.57%	0.64%
75%	3.61%	8.23%	8.2%	6.81%	11.45%	4.2%	5.82%	5.09%	3.97%	6.56%	5.64%
max	12.82%	54.12%	35.7%	47.12%	78.47%	18.17%	30.15%	24.84%	14.78%	73.14%	27.42%
SKEW	-0.61	0.52	0.24	0.76	0.59	0.04	0.06	0.19	-0.17	0.43	-0.02
KURT	1.33	2.74	0.58	3.12	4.78	0.82	1.91	0.97	0.88	8.23	0.31

	C	GS	AAPL	ACN	IBM	AMD	INTC	NVDA	COF	SCHW	INTU
count	241	241	241	241	241	241	241	241	241	241	241
mean	0.15%	1.2%	2.96%	1.4%	0.84%	2.45%	0.59%	3.86%	0.96%	1.32%	1.69%
std	12.34%	8.61%	9.37%	6.3%	6.07%	17.0%	8.07%	13.86%	10.15%	8.74%	6.89%
min	-57.75%	-27.5%	-32.96%	-14.49%	-23.66%	-41.05%	-31.02%	-38.89%	-50.33%	-32.78%	-20.72%
25%	-5.83%	-4.89%	-3.12%	-2.6%	-2.19%	-7.86%	-4.02%	-3.76%	-4.2%	-3.8%	-2.91%
50%	0.57%	1.54%	3.27%	1.94%	0.92%	0.43%	1.25%	3.35%	1.58%	0.99%	1.85%
75%	6.35%	6.56%	9.43%	5.33%	4.74%	13.76%	5.39%	12.2%	5.74%	6.86%	6.41%
max	68.67%	23.41%	35.26%	16.09%	18.25%	52.39%	31.05%	55.32%	46.27%	22.21%	18.55%
SKEW	0.25	-0.06	-0.19	-0.24	-0.38	0.32	-0.38	0.16	-0.1	-0.3	-0.19
KURT	7.95	0.42	1.17	-0.11	1.48	0.05	1.86	0.79	6.11	1.02	0.29

Based on the analysis, it is evident that lower-frequency data tends to make the distribution of returns more symmetric, as indicated by the reduction in the absolute values of skewness. This suggests that aggregating data over longer time intervals helps mitigate some of the asymmetries often observed in high-frequency return distributions. However, despite this improvement in symmetry, most stocks still fail to satisfy the condition of normal kurtosis. This implies that the return distributions exhibit either fatter tails (leptokurtic) or lighter tails (platykurtic) compared to a normal distribution.

Furthermore, when comparing data of different frequencies, it is notable that lower-frequency data tends to exhibit higher volatility, as reflected in the higher standard deviation of returns. This increase in volatility at lower frequencies may seem counterintuitive at first, but it can be attributed to the aggregation of daily fluctuations over longer periods, which amplifies the variability of returns. In contrast, high-frequency data often shows lower volatility due to the averaging effect of frequent observations, but it may also mask underlying trends and extreme events that become more apparent over longer time horizons.

Based on Figure 1, Nvidia, Apple, and Netflix exhibit the highest cumulative returns over the observed period, significantly outperforming others. In contrast, the SPX index demonstrates a moderate level of return. Financial stocks, however, show relatively weaker performance, with Citigroup standing out as the worst-performing stock during this timeframe.

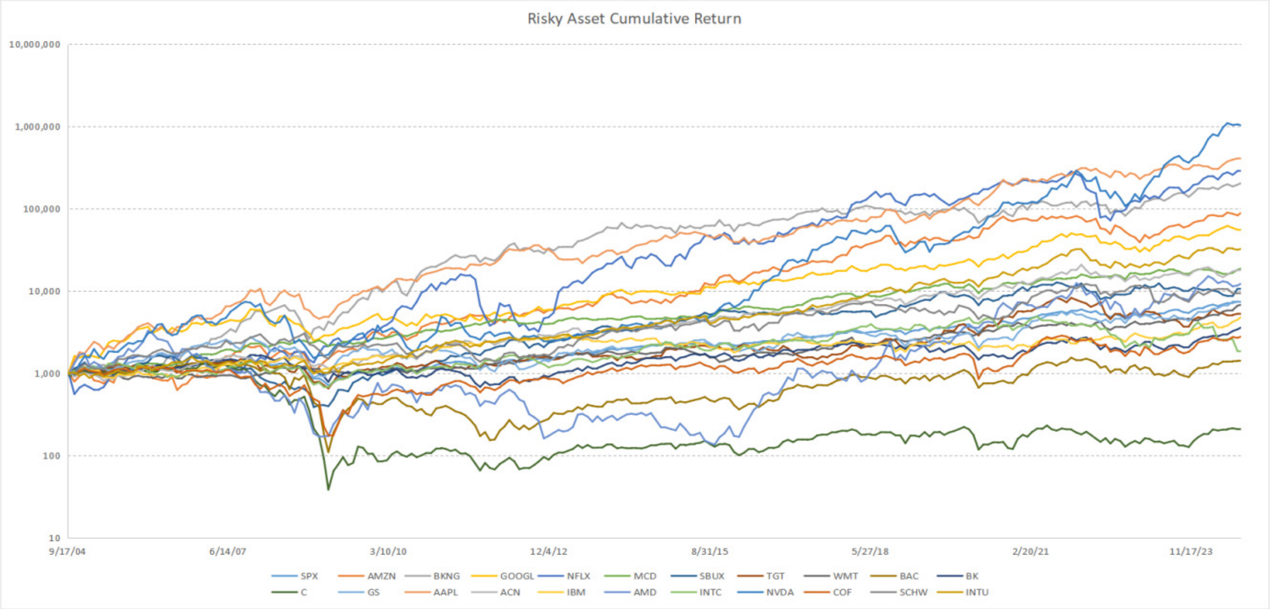

Figure 1. Risky Asset Cumulative Return

3.3 Normality Assumptions Validation

To check the assumption of normality, we first use histograms on SPX data to compare it with normal distribution. By plotting the theoretical normal distribution of the data, which is $N(E(R_P),Var(R_P))$, the gap between actual data distribution and normal distribution can be clearly observed.

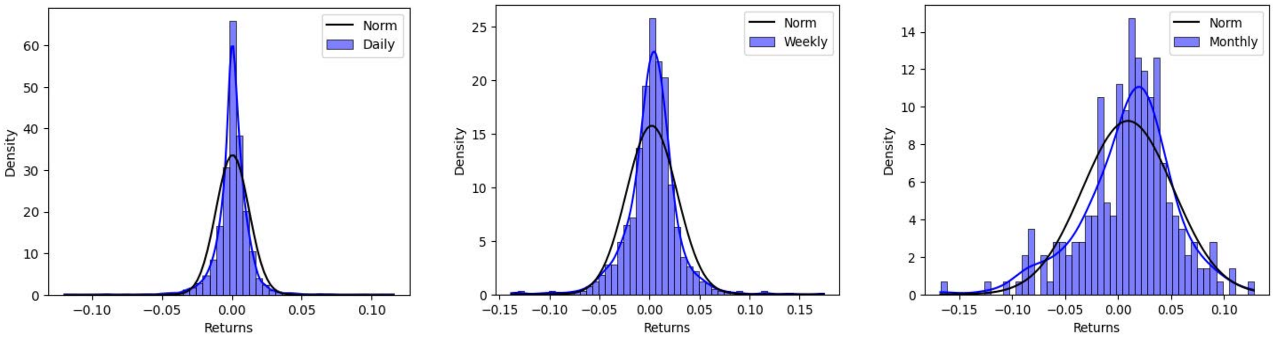

Figure 2. Comparison of Normality with SPX return over different frequencies

The lower frequency shows a better fit of normality, as monthly distribution is closer the the density of normal, while daily distribution has a thinner tail. Therefore, monthly data may indeed mitigate the violation of normal distribution. To further analyze the effect, we can also look into QQ-plot, which is common in OLS regression to check normality. As shown in Figure 3.

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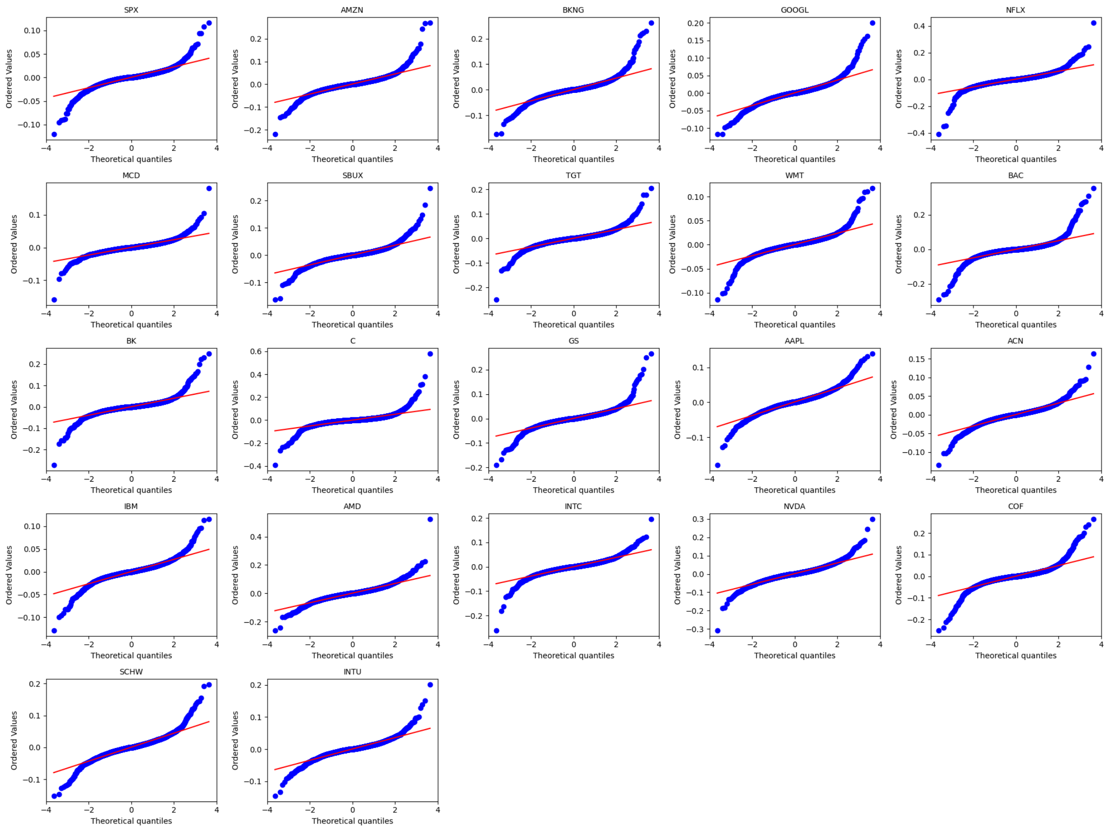

Figure 3.1. Daily return QQ-plot

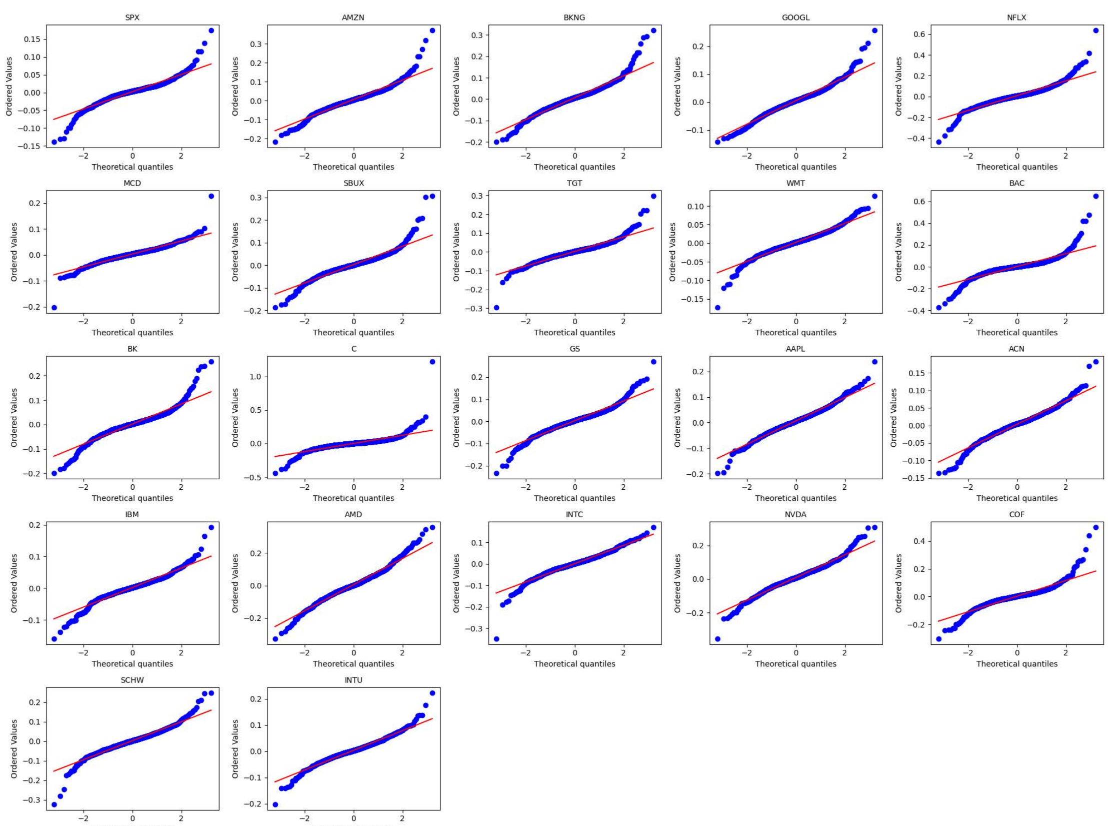

Figure 3.2. Weekly return QQ-plot

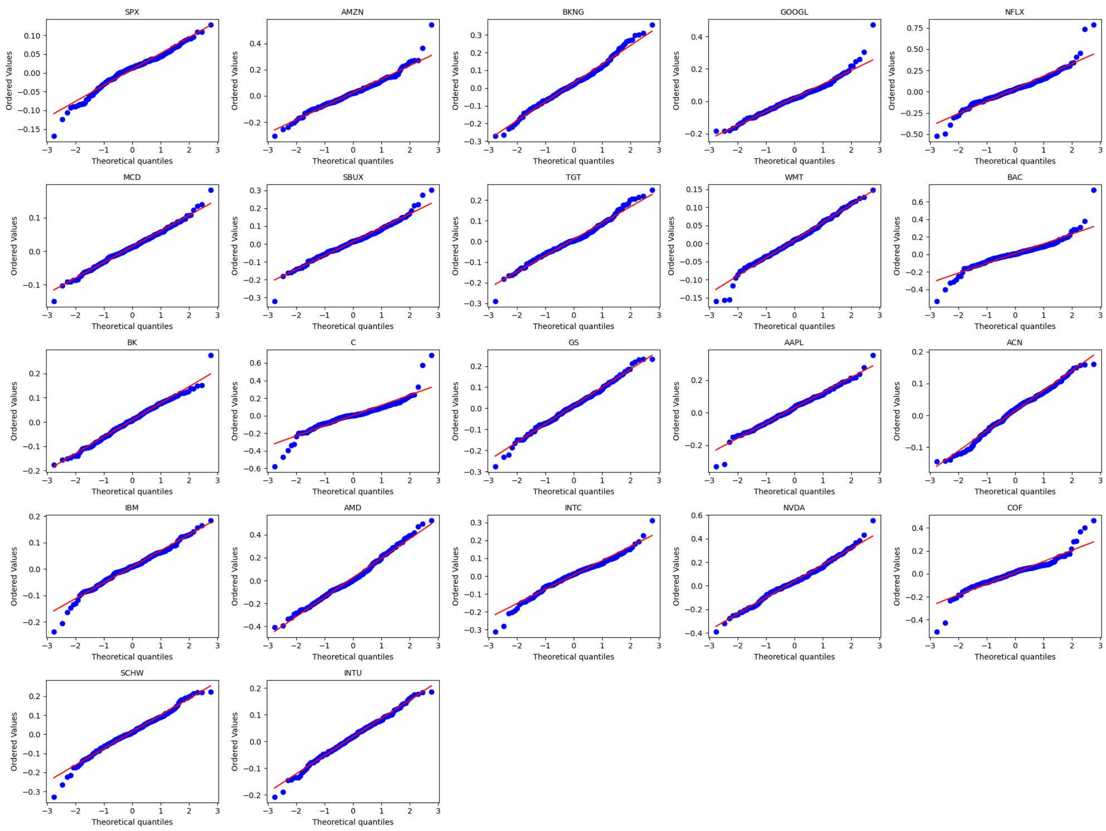

Figure 3.3. Monthly return QQ-plot

For the daily return data, the deviations from the red reference line at the tails indicate the presence of fat tails and extreme values, suggesting that daily returns exhibit non-normal characteristics with significant skewness and kurtosis. The weekly return, shows a reduction in these deviations, indicating a moderate convergence toward normality. However, slight tail deviations persist, reflecting the continued presence of some extreme values. Finally, monthly returns, demonstrates the closest alignment with the normal distribution, as the Q-Q plots align more closely with the red line. This observation suggests that the distribution of returns becomes more Gaussian-like as the return frequency lengthens, consistent with the Central Limit Theorem. These results indicate that risk modeling and return assumptions should consider non-normal behavior at higher frequencies, particularly for daily data. But monthly return indeed works for our model.

3.4 Correlation

To better analyze and to prepare for the MM and IM model, the relation of stocks with SPX is considered and their inner relations are analyzed based on our results. Here a simple OLS regression and exploratory data analysis are made to better understand the correlation of these risky instruments in Table 3.

Table 3. Regression Results of 22 assets

	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	BAC	BK
Average Return	9.5%	27.2%	31.7%	22.8%	40.3%	14.3%	13.9%	10.4%	9.4%	8.5%	7.6%
Annual StDev	15.0%	36.0%	36.8%	29.7%	52.4%	16.1%	27.1%	27.4%	17.1%	40.5%	23.8%
beta	100.0%	120.6%	120.6%	104.1%	105.8%	57.5%	100.5%	97.1%	40.3%	172.2%	101.2%
alpha	0.0%	15.7%	20.2%	12.8%	30.2%	8.8%	4.3%	1.2%	5.6%	-7.9%	-2.1%

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residual Stdev	0.0%	31.2%	32.0%	25.3%	49.9%	13.6%	22.5%	23.2%	16.0%	31.3%	18.3%
	C	GS	AAPL	ACN	IBM	AMD	INTC	NVDA	COF	SCHW	INTU
Average Return	0.2%	12.8%	33.9%	15.1%	8.4%	27.7%	5.4%	44.6%	9.9%	14.2%	18.7%
Annual StDev	42.7%	29.8%	32.5%	21.9%	21.0%	58.9%	28.0%	48.0%	35.2%	30.3%	23.9%
beta	194.5%	138.7%	123.8%	104.2%	81.0%	220.7%	102.5%	175.9%	150.7%	120.7%	100.0%
alpha	-18.3%	-0.5%	22.1%	5.2%	0.7%	6.7%	-4.4%	27.9%	-4.4%	2.7%	9.1%
residual Stdev	31.3%	21.4%	26.7%	15.3%	17.1%	48.8%	23.4%	40.1%	27.0%	24.3%	18.7%

To better examine the assets. Heatmap of correlation is applied as shown in Figures 4. It becomes evident that stocks within the same industry exhibit high correlations, particularly within the financial sector. Notably, the correlation between BAC and C is the strongest, with a coefficient of 0.84, indicating a strong linear relationship. This pattern is further supported by the data presented in Table 1, which highlights the similar stock price trends of these pairs, reinforcing the strength of their correlation. In contrast, stocks from different industries, demonstrate much lower correlation coefficients. Overall, while most stock pairs show relatively weak correlations, with a few notable exceptions, it is clear that the assumption of non-correlation does not always hold in practice.

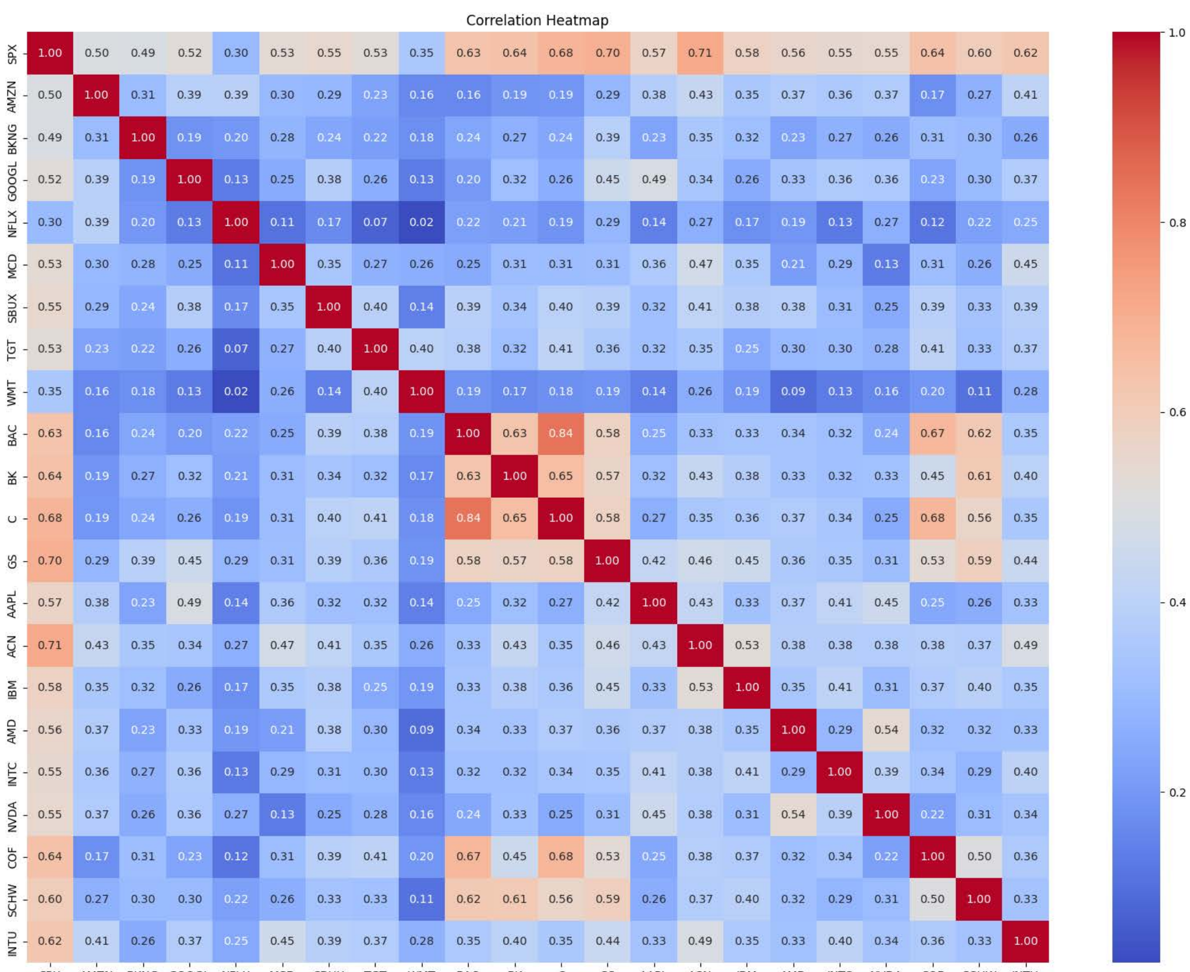

Figure 4. Heatmap of correlation

4. PORTFOLIO CONSTRUCTION & COMPARISON

We employ SolverTable to compute the permissible portfolio regions for both the Markowitz Model (MM) and the Single Index Model (IM) under various constraints. These regions include the efficient frontier, the inefficient frontier, and the minimum variance frontier, which collectively define the set of feasible portfolios based on the given constraints.

For each model, we identify two critical points on the efficient frontier:

1. Global Minimum Variance Portfolio (GMVP): This portfolio represents the point with the lowest possible risk (variance) on the efficient frontier. It is the optimal choice for risk-averse investors seeking to minimize portfolio volatility.

2. Maximal Sharpe Ratio Portfolio (Efficient Risky Portfolio): This portfolio corresponds to the point where the Capital Allocation Line (CAL) is tangent to the efficient frontier. It offers the highest risk-adjusted return, as measured by the Sharpe Ratio, and is optimal for investors seeking to maximize returns per unit of risk.

3. The Capital Allocation Line (CAL): a straight line that represents the combination of the risk-free asset and the optimal risky portfolio (Maximal Sharpe Ratio Portfolio). The slope of the CAL, which is the Sharpe Ratio, quantifies the trade-off between risk and return, providing a measure of portfolio efficiency.

By comparing the results of the MM and IM models, we analyze the differences in the permissible portfolio regions, the location of the GMVP and Maximal Sharpe Ratio Portfolio, and the shape of the CAL. This comparison allows us to evaluate the impact of model assumptions and constraints on portfolio optimization outcomes. Specifically, we assess how the MM's reliance on full covariance matrices contrasts with the IM's simplification through the use of a single market index, and how these differences influence the construction of efficient portfolios under varying constraints.

4.1 The regulation T by FINRA

In the first constraint, our additional optimization constraint included in this model is designed to simulate Regulation T set forth by the Financial Industry Regulatory Authority (FINRA), which governs margin accounts. According to Regulation T, broker-dealers are permitted to extend margin loans to customers, allowing them to hold positions in their accounts that are funded at least 50% by their own equity. This means that for any margin position, the customer must contribute a minimum of 50% of the total value with their own funds, while the remaining portion may be borrowed from the broker-dealer. The constraint is mathematically represented as: $\sum_{i=1}^{n}|w_i| \leqslant 2$

. This constraint ensures that the total magnitude of the portfolio's leveraged positions does not exceed twice the customer's equity, aligning with the regulatory framework to prevent excessive leverage and reduce systemic risk in the financial system.

The problem is solved as following Table 4 shows. MM model has slightly better return than IM model, but riskier, which leads to worse Sharpe ratio.

Table 4. MM and IM result of Constraint 1

MM (Prob. 1):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	BAC	BK	Return	StDev	Sharpe
MinVar	83.23%	-1.76%	-3.67%	0.71%	1.95%	21.05%	0.85%	0.00%	23.05%	-1.07%	9.64%	7.10%	10.53%	0.674
MaxSharpe	-8.01%	0.00%	14.53%	11.30%	10.42%	40.80%	1.02%	0.00%	20.64%	4.65%	0.00%	30.69%	17.76%	1.728
	C	GS	AAPL	ACN	IBM	AMD	INTC	NVDA	COF	SCHW	INTU			
MinVar	-12.06%	-4.75%	-3.47%	-9.69%	7.83%	-3.19%	1.20%	-3.72%	-2.05%	0.49%	-4.57%			
MaxSharpe	-16.68%	-8.00%	19.90%	0.00%	0.00%	-2.29%	-15.02%	11.32%	1.59%	6.21%	7.62%			
IM (Prob. 1):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	BAC	BK	Return	StDev	Sharpe
MinVar	97.46%	-1.80%	-1.70%	-0.19%	-0.12%	22.62%	0.00%	0.53%	23.01%	-6.97%	0.00%	7.08%	9.86%	0.719
MaxSharpe	0.00%	9.32%	12.74%	11.14%	8.91%	35.33%	2.64%	0.23%	19.45%	-9.86%	-2.36%	32.08%	16.57%	1.936
	C	GS	AAPL	ACN	IBM	AMD	INTC	NVDA	COF	SCHW	INTU			
MinVar	-9.21%	-7.72%	-2.91%	-0.59%	6.38%	-4.88%	0.00%	-4.47%	-6.46%	-2.98%	0.00%			
MaxSharpe	-19.89%	-3.41%	20.44%	4.94%	1.54%	-0.27%	-5.52%	10.35%	-8.50%	-0.18%	12.98%			

The table reveals different weight allocations for the stocks, with a marked difference between stocks in various sectors. The weight difference can be better viewed in Figure 5. For example, stocks such as SPX, AMZN, and BKNG are assigned relatively high weights in comparison to others like INTC and NVDA, indicating a preference for companies with strong historical performance or stability in the portfolio. The diversification across various industries, including technology, finance, and consumer goods, appears to be aimed at reducing sector-specific risks.

In terms of risk, the portfolio likely experiences some lev-

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el of diversification benefit, as evidenced by the relatively balanced allocation of assets across sectors. However, the high concentration in certain stocks could increase exposure to specific market risks, particularly if any of these key stocks underperform. The optimization process appears to favor a balance between risk and return, though it is evident that the potential for high returns is somewhat contingent on the performance of a select group of stocks, especially those within the tech and financial sectors.

Overall, while the portfolio weighted results reflect a strategy that prioritizes stable and high-growth stocks, which could result in favorable returns assuming these stocks maintain their upward trajectory in the market.

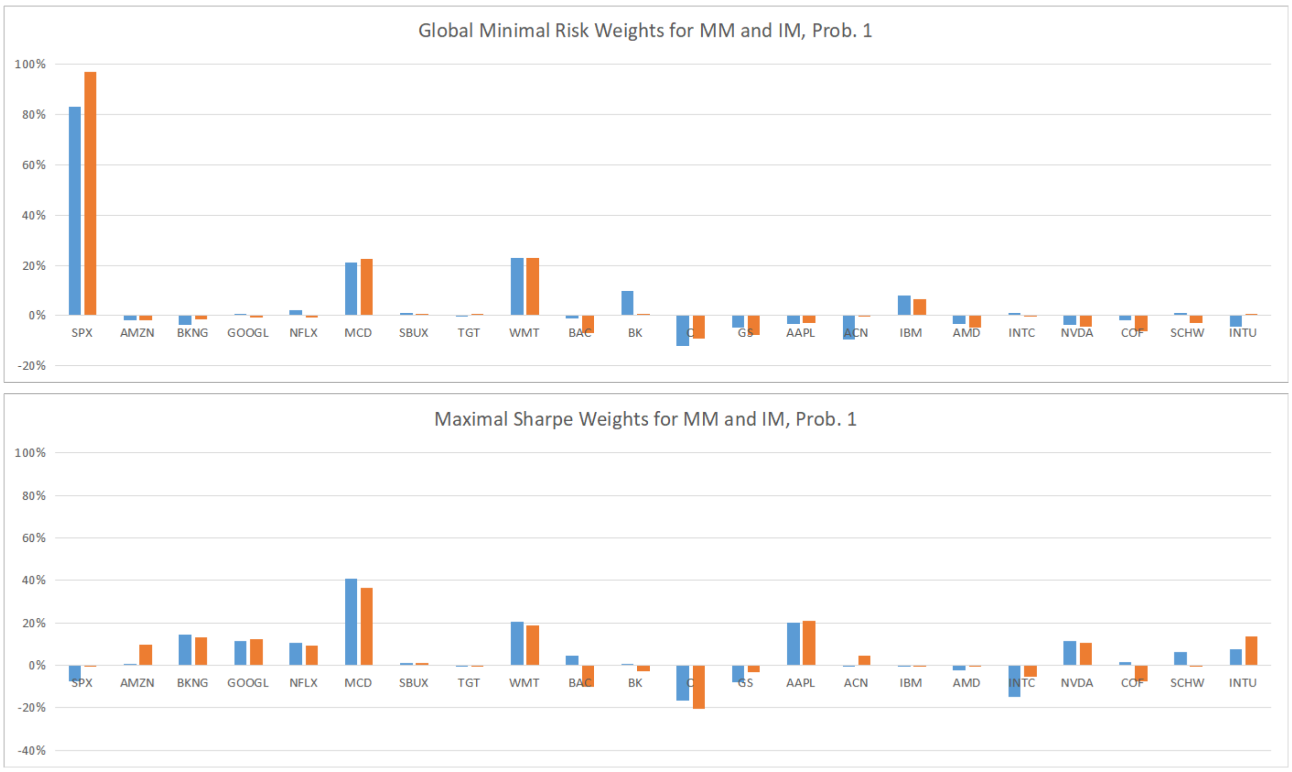

Figure 5. MM and IM weight of Constraint 1

It's also notable that when it comes to maximize Sharpe ratio, the weights tend to be reduced, which is exactly a reflection of risk balancing. In this case, the portfolios are more diversified compared with minimizing the risk.

To understand their relationship, we plot the regions and CAL line in Figure 6. The feasible region is delineated by the various curves and points shown within the plot. Here IM feasible region is larger than MM model, which means more choices, and so the tangent line CAL has a steeper slope than MM model.

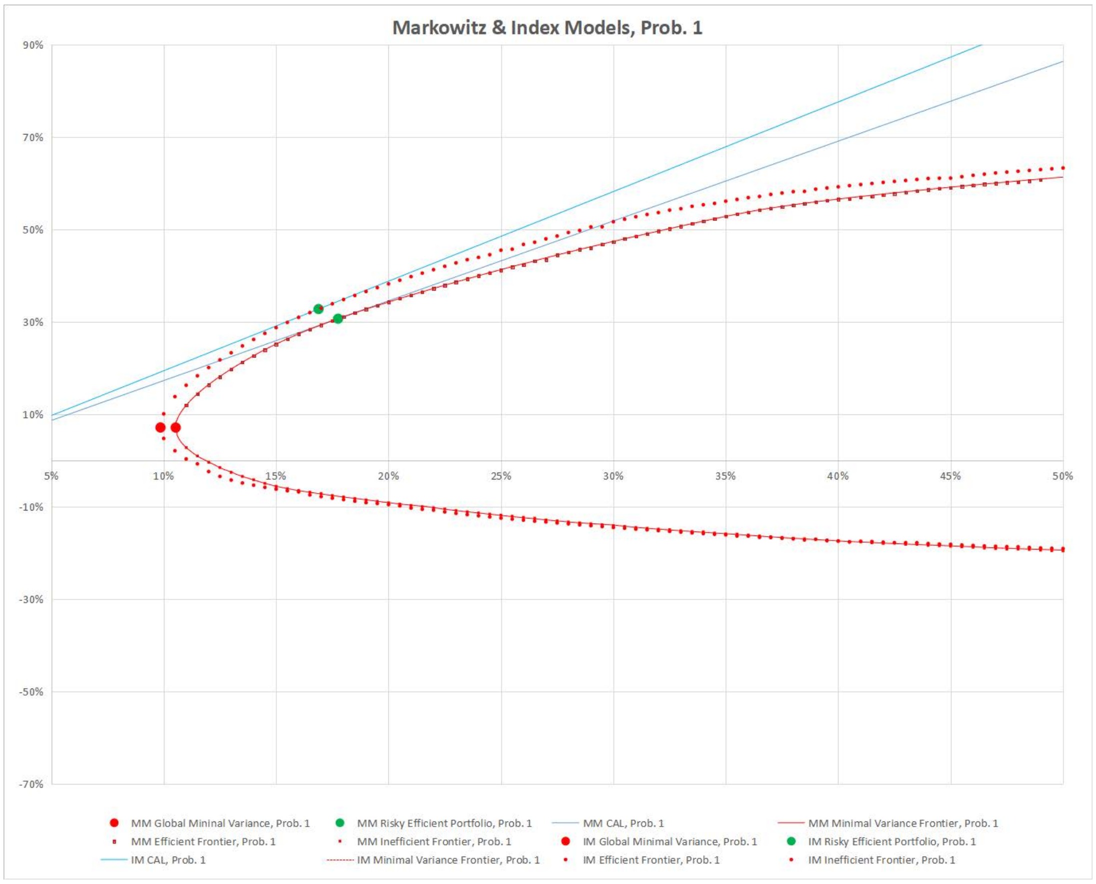

Figure 6. MM and IM feasible regions and CAL

But how will the margin affect our portfolios? Take MM model for example, as the requirement changes, if $\sum_{i=1}^{n}|w_i| \leqslant k$, k moves from 1 to 2, investors will face lower risk to higher risk as they can increase the leverage. By making sensitivity analysis, we can see the effect in Figure 7.

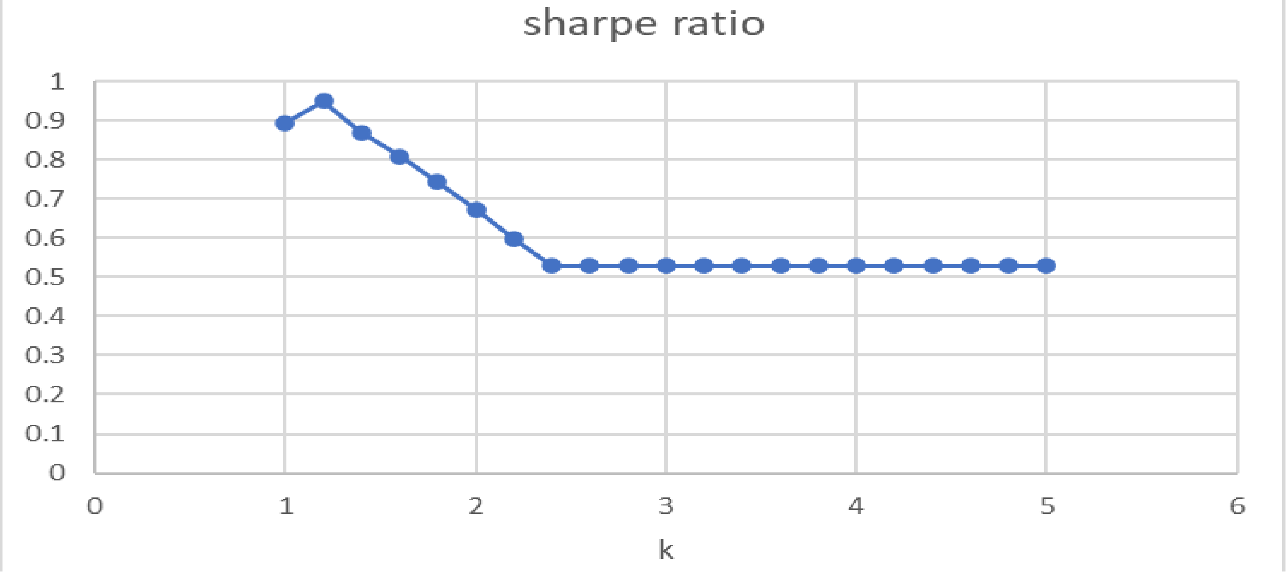

Figure 7. Sharpe ratio when k changes

As the k moves, it is evident that the Sharpe ratio initially increases, followed by a decline, eventually stabilizing at

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a lower level. This pattern may be attributed to the reduction in risk due to a stricter margin requirement, which limits potential losses compared to earlier scenarios. Consequently, we can infer that if the government were to relax margin regulations, investors might be exposed to excessive risk, ultimately leading to suboptimal outcomes. The increased leverage and risk-taking behavior encouraged by more lenient margin requirements could result in greater volatility and potential long-term financial harm.

4.2 Long-Only portfolios and L1 regularization

Now we focus on constraint 3 and 4, which is a free problem and a long-only problems. In the second case, which is for mutual fund, the weights are significantly lower and sparse as shown in Figure 8.

This can be explained by L1 regularization, which is common in Machine Learning. L1 regularization leads to sparsity because of its unique penalty structure, which encourages some coefficients to shrink to exactly zero.

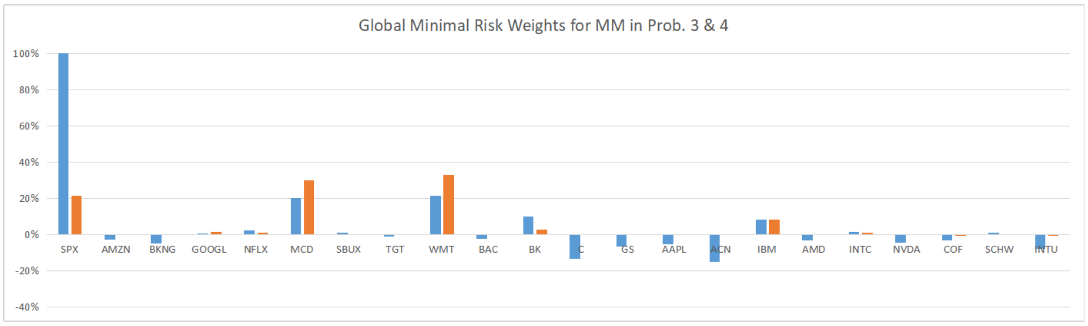

Figure 8. Constraint 3 and 4 weights comparison

By examining the feasible region in Figure 9, it becomes evident that Constraint 4 reduces the size of the feasible region. This restriction effectively narrows the set of possible portfolios but ensures, at least theoretically, that the portfolio will achieve a return greater than zero. By imposing stricter bounds, Constraint 4 eliminates portfolios with potential negative returns, thus enhancing the likelihood of achieving positive performance while maintaining a more conservative risk-return profile.

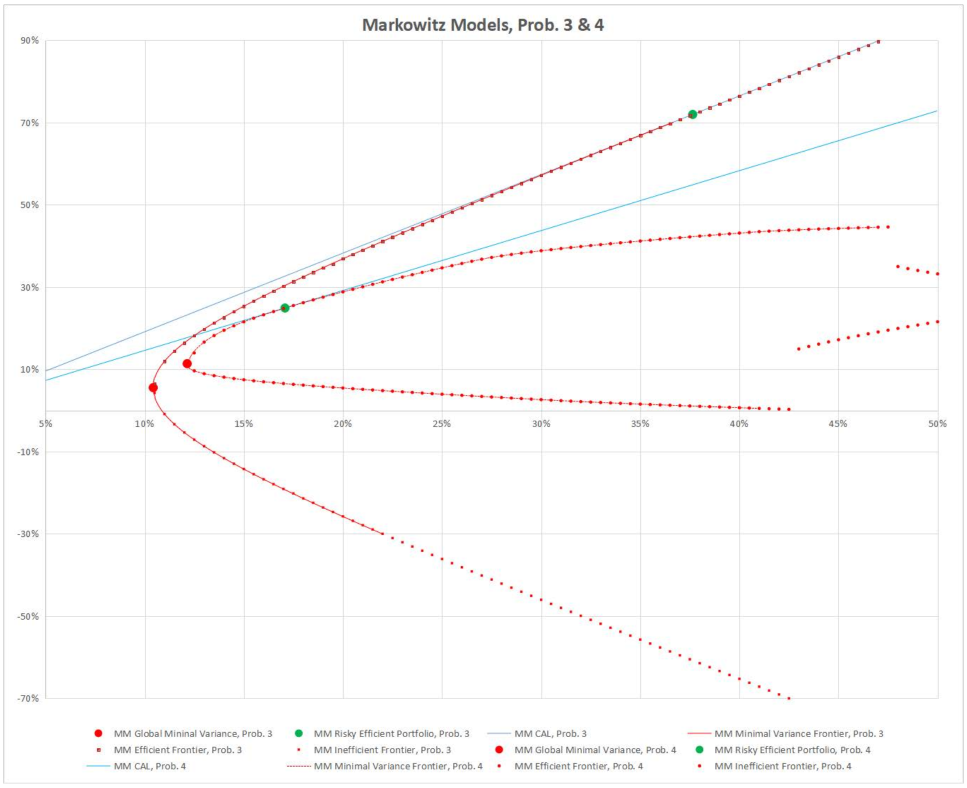

Figure 9. Constraint 3 and 4 feasible region

4.3 With or Without Index

When the S&P 500 (SPX) is excluded from the portfolio construction process, a significant effect on portfolio performance is observed, particularly in comparison to other constraints. As shown in Table 5, both the Markowitz Model (MM) and the Single Index Model (IM) exhibit relatively higher returns and improved Sharpe ratios when the SPX is set to zero. This finding suggests that the exclusion of the broad market index from the portfolio has a meaningful impact on the risk-return profile of the optimized portfolios.

Table 5. Constraint 5 compared with other constraints

MM (Prob. 1):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	83.23%	-1.76%	-3.67%	0.71%	1.95%	21.05%	0.85%	0.00%	23.05%	7.10%	10.53%	0.674
MaxSharpe	-8.01%	0.00%	14.53%	11.30%	10.42%	40.80%	1.02%	0.00%	20.64%	30.69%	17.76%	1.728
MM (Prob. 2):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	100.00%	-2.68%	-4.94%	0.94%	2.42%	20.55%	1.35%	-0.86%	21.75%	5.62%	10.43%	0.539
MaxSharpe	-100.00%	0.91%	24.35%	21.28%	13.82%	60.55%	6.09%	-3.38%	32.60%	44.59%	23.88%	1.867
MM (Prob. 3):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	101.90%	-2.69%	-5.00%	0.85%	2.40%	20.24%	1.30%	-0.89%	21.49%	5.52%	10.43%	0.529
MaxSharpe	-318.23%	3.85%	44.62%	37.81%	21.06%	97.88%	12.04%	-2.29%	53.55%	71.93%	37.65%	1.911
MM (Prob. 4):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	21.86%	0.00%	0.00%	1.63%	1.22%	30.25%	0.00%	0.00%	33.13%	11.36%	12.14%	0.935
MaxSharpe	0.00%	0.00%	12.36%	5.46%	10.33%	30.46%	0.00%	0.00%	15.40%	24.88%	17.08%	1.456
MM (Prob. 5):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	0.00%	-1.77%	-0.90%	5.97%	3.40%	34.79%	4.43%	1.38%	35.47%	11.32%	11.68%	0.969
MaxSharpe	0.00%	-0.10%	19.11%	15.67%	12.31%	45.63%	3.10%	-5.22%	19.83%	37.32%	21.21%	1.759
IM (Prob. 1):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	97.46%	-1.80%	-1.70%	-0.19%	-0.12%	22.62%	0.00%	0.53%	23.01%	7.08%	9.86%	0.719
MaxSharpe	0.00%	9.32%	12.74%	11.14%	8.91%	35.33%	2.64%	0.23%	19.45%	32.08%	16.57%	1.936
IM (Prob. 2):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	100.00%	-1.97%	-1.87%	-0.50%	-0.19%	22.70%	0.05%	0.67%	22.98%	6.85%	9.85%	0.696
MaxSharpe	-100.00%	17.39%	21.62%	21.68%	13.68%	54.83%	8.01%	0.84%	26.00%	49.40%	23.98%	2.060
IM (Prob. 3):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	103.34%	-2.06%	-1.94%	-0.61%	-0.22%	22.26%	-0.11%	0.52%	22.65%	6.69%	9.85%	0.679
MaxSharpe	-199.42%	23.37%	28.55%	28.99%	17.56%	69.26%	12.31%	3.09%	31.50%	62.34%	30.07%	2.073
IM (Prob. 4):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	23.48%	0.00%	0.00%	0.00%	0.00%	32.95%	0.00%	0.76%	33.52%	10.97%	11.98%	0.916
MaxSharpe	0.00%	6.45%	11.47%	7.07%	10.26%	25.05%	0.00%	0.00%	9.63%	26.64%	18.04%	1.476
IM (Prob. 5):	SPX	AMZN	BKNG	GOOGL	NFLX	MCD	SBUX	TGT	WMT	Return	StDev	Sharpe
MinVar	0.00%	0.45%	0.42%	3.19%	0.76%	35.71%	4.72%	5.07%	32.44%	11.80%	11.06%	1.067
MaxSharpe	0.00%	13.37%	17.25%	16.37%	11.41%	41.15%	3.49%	-2.61%	18.28%	40.86%	20.58%	1.985

The potential reasons behind this phenomenon and its implications for portfolio management can be attributed to the following reasons.

4.3.1 Reduced Market Dependency & Enhanced Diversification

The SPX, as a broad market index, is highly correlated with many individual stocks, particularly those in the technology and financial sectors. By excluding the SPX, the portfolio is no longer tied to the systemic risk associated with the broader market, allowing for greater flexibility in asset allocation. This reduction in market dependency may enable the optimization models to identify alternative combinations of assets that offer better risk-adjusted returns. In essence, the exclusion of the SPX forces the models to rely more heavily on the idiosyncratic risk and return characteristics of individual stocks, potentially leading to a more diversified and efficient portfolio.

4.3.2 Reallocation of Weights to Higher-Alpha Assets

Another contributing factor is the reallocation of portfolio weights to assets with higher alpha when the SPX is excluded. The SPX, as a market proxy, often carries a lower expected return compared to high-growth individual stocks, such as those in the technology sector. When the SPX is removed, the optimization models may allocate more weight to stocks with higher historical returns and stronger growth potential, such as Nvidia, Apple, or Netflix. This reallocation can lead to an increase in the overall portfolio return. Additionally, the improved Sharpe ratio indicates that the additional return is achieved without a proportional increase in risk, suggesting that the models are effectively balancing the trade-off between risk and return.

4.3.3 Impact on Covariance Structure and Risk Estimation

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The exclusion of the SPX also alters the covariance structure of the portfolio, which plays a critical role in risk estimation. In the Markowitz Model, the covariance matrix captures the relationships between asset returns, and the SPX, as a highly correlated asset, can dominate this matrix. By removing the SPX, the covariance matrix becomes less influenced by market-wide movements, allowing the model to better capture the unique risk-return profiles of individual assets. This adjustment may lead to a more accurate estimation of portfolio risk, enabling the models to construct portfolios that are better optimized for specific risk-return objectives. In the Single Index Model, the exclusion of the SPX removes the primary market factor, forcing the model to rely on other sources of systematic risk, which may result in a more nuanced and efficient portfolio construction.

In summary, the exclusion of the SPX from portfolio construction leads to higher returns and improved Sharpe ratios in both the MM and IM models, primarily due to reduced market dependency, reallocation of weights to higher-alpha assets, and adjustments to the covariance structure. These findings highlight the importance of carefully considering the role of broad market indices in portfolio optimization and suggest that their exclusion can be a viable strategy for enhancing portfolio performance under certain conditions. Since we are entering an era where ETF, market portfolios are highly valued, this results may serve as a different angle for investors to reconsider the role of market portfolio instead of buying them recklessly.

5. Conclusion

This study has provided a comprehensive comparative analysis of portfolio optimization under various constraints, focusing on the Markowitz Model (MM) and the Single Index Model (IM). Our findings indicate that the MM model, with its reliance on full covariance matrices, offers a more favorable risk-return trade-off compared to the IM model, particularly when regulatory constraints such as Regulation T are imposed. The sensitivity analysis revealed that stricter margin requirements, as mandated by Regulation T, can significantly influence portfolio risk and return profiles. Specifically, we observed that as leverage constraints are relaxed, investors face higher risks, which can lead to suboptimal outcomes. This underscores the importance of regulatory frameworks in maintaining financial stability and protecting investors from excessive risk-taking. Policymakers should consider these findings when evaluating the potential impact of changes to margin regulations, as overly lenient rules could exacerbate systemic risks in the financial markets.

For Fund of Funds (FOF) investors, our analysis highlights the critical role of diversification and risk management in constructing portfolios that align with regulatory requirements and investor risk preferences. The long-only constraint, commonly applied in mutual funds, was shown to reduce the feasible region of portfolios but also ensured more conservative risk-return profiles. This is particularly relevant for FOF managers, who must balance the need for diversification across multiple asset classes with the constraints imposed by regulatory frameworks. Our results suggest that FOF managers should prioritize robust portfolio optimization techniques, such as the MM model, to achieve better risk-adjusted returns while adhering to regulatory guidelines. Additionally, the use of L1 regularization in portfolio construction, which promotes sparsity and reduces over-concentration in specific assets, could be a valuable tool for FOF managers seeking to enhance portfolio stability.

Looking ahead, the integration of deep learning techniques into portfolio optimization presents a promising avenue for future research. While traditional models like MM and IM have laid the foundation for modern portfolio theory, they often struggle to capture the complex, non-linear relationships inherent in financial markets. Deep learning models, with their ability to process vast amounts of data and identify intricate patterns, could offer significant improvements in predicting asset returns and optimizing portfolio weights. Future studies could explore the application of deep learning frameworks, such as neural networks and reinforcement learning, to enhance the accuracy of risk and return estimates. Moreover, the combination of deep learning with traditional optimization methods could lead to more adaptive and resilient portfolio strategies, particularly in volatile market conditions. As the field of quantitative finance continues to evolve, the integration of advanced computational techniques with established financial theories will likely play a pivotal role in shaping the future of portfolio management.

References

[1] Markowitz, H. (1952). Portfolio selection. The Journal of Finance, 7(1), 77-91

[2] Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. The Journal of Finance, 19(3), 425-442.

[3] Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. Journal of Financial Economics, 33(1), 3-56.

[4] Carhart, M. M. (1997). On persistence in mutual fund performance. The Journal of Finance, 52(1), 57-82.

[5] Ledoit, O., & Wolf, M. (2003). Improved estimation of the covariance matrix of stock returns with an application to

portfolio selection. Journal of Empirical Finance, 10(5), 603-621.
[6] DeMiguel, V., Garlappi, L., & Uppal, R. (2009). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? The Review of Financial Studies, 22(5), 1915-1953.
[7] Black, F., & Litterman, R. (1992). Global portfolio optimization. Financial Analysts Journal, 48(5), 28-43.
[8] Michaud, R. O., & Michaud, R. O. (2008). Efficient asset management: A practical guide to stock portfolio optimization and asset allocation. Oxford University Press.
[9] Bodie, Z., Kane, A., & Marcus, A. J. (2014). Investments (10th ed.). McGraw-Hill Education.
[10] Fabozzi, F. J., Kolm, P. N., Pachamanova, D. A., & Focardi, S. M. (2007). Robust portfolio optimization and management. John Wiley & Sons.