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Portfolio Optimization Analysis with Markowitz and Index Models in Capital Markets

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Abstract:

This project examines the outcomes of portfolio optimization using the Markowitz Model and the Index Model, applied to more than 20 years of historical data from ten prominent U.S. stocks and one broad equity index. The S&P 500 index serves as the benchmark equity index, while the one-month Fed Funds rate is utilized as the proxy for the risk-free rate.

Keywords: Markowitz Model, Index Model, Portfolio Optimization

1. Introduction

This paper aims to analyze and compare the Markowitz Model (MM) and the Index Model (IM) using historical daily total return data for the S&P 500 index and a selection of ten U.S. stocks. The study focuses on calculating the necessary optimization inputs for both models and identifying the regions of permissible portfolios under five additional constraints. The analysis is structured to provide a comprehensive understanding of Modern Portfolio Theory, particularly through the lens of Harry Markowitz's Optimal Portfolio Selection Model and William Sharpe's Index Model as its simplified approximation. To understand the key concepts of Modern Portfolio Theory (MPT), focusing on Harry Markowitz's Optimal Portfolio Selection Model and William Sharpe's Index Model as its simplified approximation.

The dataset consists of 20 years of daily total return data, which is aggregated into monthly observations to mitigate non-Gaussian effects and improve the reliability of the optimization inputs. The paper is organized as follows: Section 2 introduces the companies and their stock price data; Section 3 provides an overview of the Markowitz Model, the Index Model, the additional constraints and other analytical tools used in the study; Section 4 discusses the data processing steps, including the aggregation of daily data and computation of optimization inputs; and Section 5 compares the results of MM and IM under five additional constraints. The final section presents the results and conclusions, highlighting the practical implications for portfolio optimization. The primary goals of this study are:

1. To construct optimal portfolios using both models and additional constraints, based on historical stock prices and a broad market index.
2. To compare the performance of the two models and illustrate that the Index Model serves as a good approximation of the full Markowitz Model.
3. To analyze and compare the impact of the five additional constraints on portfolio optimization within each model.

This paper provides a comprehensive analysis of the Markowitz and Index Models, shedding light on their strengths and weaknesses in the context of portfolio optimization. The results are expected to offer valuable insights for investors aiming to optimize their portfolios, balancing risk reduction with the maximization of returns.

2. Background of the Selected Companies

The data used in this article is sourced from Yahoo! Finance. It includes daily stock prices for 21 companies from September 17, 2004, to September 20, 2024, along with the corresponding S&P 500 index and the risk-free interest rate (the one-month federal funds rate). The selected companies primarily belong to four sectors: Technology, Financial Services, Materials and Industrials. The specific details are as follows:

1. SPX (S&P 500 Index)

The S&P 500 is a stock market index representing 500 of the largest publicly traded companies in the U.S. It serves as a key benchmark for the overall health of the U.S. stock market and is widely used by investors to track the performance of the market.

2. JPM (JPMorgan Chase)

JPMorgan Chase is one of the largest financial services institutions in the world, offering a comprehensive range of services including investment banking, financial services for consumers and businesses, asset management, and private equity. It operates globally and provides services like mergers and acquisitions, financial advisory, and treasury services.

3. MS (Morgan Stanley)

Morgan Stanley is a multinational investment bank and financial services company. It provides capital markets, investment banking, wealth management, and investment management services to individual, corporate, and institutional clients. Known for its strong presence in global financial markets, it is a leader in trading, investment strategies, and advisory.

4. USB (U.S. Bancorp)

U.S. Bancorp is the parent company of U.S. Bank, one of the largest commercial banks in the U.S. It provides a wide range of banking products and services, including retail banking, asset management, private banking, corporate banking, and treasury services. The company serves individuals, businesses, and institutions across the U.S.

5. WFC (Wells Fargo)

Wells Fargo is a multinational financial services company providing banking, investment, mortgage, and consumer and commercial finance services. Known for its retail banking services, Wells Fargo offers checking and savings accounts, mortgages, auto loans, credit cards, and investment management services to individuals and businesses.

6. EMR (Emerson Electric)

Emerson Electric is a global technology and engineering company. It provides innovative solutions in areas such as automation, climate technologies, industrial software, and power systems. Emerson serves industries like manufacturing, food and beverage, energy, and chemical processing, focusing on efficiency and sustainability in its products and services.

7. HON (Honeywell)

Honeywell is a multinational conglomerate that designs and manufactures products in several sectors, including aerospace, building technologies, chemicals, and materials. Its products include air and water purification systems, home automation solutions, industrial control systems, and aerospace parts. The company is also involved in developing technologies for the energy and defense sectors.

8. CAT (Caterpillar)

Caterpillar is a leading manufacturer of construction and mining equipment, diesel engines, and industrial gas turbines. It serves industries like construction, mining, oil and gas, and forestry, providing heavy equipment for building, excavation, and resource extraction. Caterpillar is also involved in offering financial products and services to customers.

9. DE (John Deere)

John Deere is an American manufacturer of agricultural, construction, and forestry machinery. The company is best known for its agricultural equipment, such as tractors, harvesters, and plows. It also manufactures construction machinery like excavators and bulldozers. John Deere also offers financial services to help customers finance their machinery purchases.

10. MMM (3M)

3M is a diversified technology company known for its wide range of products in fields such as healthcare, safety, consumer goods, and industrials. Its products include adhesives, coatings, medical supplies, and office products. 3M's innovation is driven by science and technology, with a strong focus on sustainability and improving daily life.

11. FDX (FedEx)

FedEx is a global logistics and package delivery company. It offers overnight shipping, freight transportation, international shipping, and e-commerce services. FedEx operates several divisions, including FedEx Express (air-based logistics), FedEx Ground (parcel shipping), and FedEx Freight (less-than-truckload freight). It is a major player in the global shipping and delivery sector.

12. UNP (Union Pacific)

Union Pacific is one of the largest freight railroad companies in the United States. It operates a vast network of rail lines that transport goods, including agricultural products, chemicals, automotive, and industrial products. Union

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Pacific plays a key role in connecting businesses with international markets through its transportation services.

13. UPS (United Parcel Service)

UPS is a global leader in package delivery and supply chain management. In addition to its well-known package delivery services, UPS offers logistics, freight transportation, and distribution services to businesses worldwide. It plays a major role in e-commerce, providing efficient delivery solutions for online retailers and customers.

14. AAPL (Apple)

Apple is one of the world's most valuable technology companies, known for its consumer electronics, software, and services. Apple's products include the iPhone, iPad, Mac computers, and Apple Watch. The company is also involved in services such as iCloud, Apple Music, and the App Store. Known for its innovation and design, Apple has a large customer base worldwide.

15. ACN (Accenture)

Accenture is a global professional services company that provides consulting, technology, and outsourcing services. Accenture helps businesses improve their operations through digital transformation, strategy consulting, and IT solutions. It has a strong presence in sectors like financial services, healthcare, and telecommunications.

16. IBM (International Business Machines)

IBM is a multinational technology company that provides hardware, software, and services in areas such as artificial intelligence, cloud computing, and enterprise software. IBM is known for its innovation in the tech sector, particularly through its Watson AI platform. The company offers solutions to businesses for data management, cybersecurity, and business analytics.

17. AMD (Advanced Micro Devices)

AMD is a semiconductor company known for its high-performance processors, graphics cards, and other computer hardware. The company competes directly with Intel in the CPU market and with NVIDIA in the GPU market. AMD's products are used in personal computers, gaming consoles, and data centers, offering high performance for computing and graphics-intensive applications.

18. INTC (Intel)

Intel is one of the world's largest semiconductor companies, primarily known for its microprocessors, which are used in computers, servers, and embedded devices. Intel is a leader in the development of computer hardware, including chips for personal computers, data centers, and IoT devices. It has a significant role in advancing computing technology and performance.

19. NVDA (NVIDIA)

NVIDIA is a leading technology company specializing in graphics processing units (GPUs) for gaming, AI, and high-performance computing. NVIDIA's GPUs are widely used in gaming, professional visualization, data centers, and deep learning applications. The company is also a leader in AI research and development, providing technologies for self-driving cars, healthcare, and more.

20. QCOM (Qualcomm)

Qualcomm is a global semiconductor and telecommunications equipment company. It is a leader in developing wireless technologies, including mobile phone chips, 5G infrastructure, and IoT solutions. Qualcomm is known for its Snapdragon processors used in mobile devices, as well as its leadership in the development and commercialization of 5G technologies.

21. TXN (Texas Instruments)

Texas Instruments designs and manufactures semiconductors and various integrated circuits. Its products are used in a wide range of applications, from consumer electronics to industrial systems. Texas Instruments is known for its analog and embedded processing solutions, which are crucial for products in automotive, communications, and industrial markets.

22. LIN (Linde)

Linde is a global industrial gases and engineering company. It supplies gases like oxygen, nitrogen, and hydrogen to various industries, including healthcare, chemicals, energy, and manufacturing. Linde's products are used in applications ranging from medical treatments to industrial processes such as welding, refrigeration, and energy production.

In conclusion, the extensive dataset, which includes the S&P 500 index and a selection of prominent U.S. stocks, offers a distinctive opportunity to explore the relationship between overall market trends and the performance of individual companies. By analyzing both market-wide sentiment and company-specific factors, this study seeks to uncover valuable insights for investment strategies and illuminate the interactions of various market forces.

3. Overview of the Markowitz Model and the Index Model

Shifting from the exploration of our dataset, we now focus on the core of our study: the two essential models that will inform our analysis. In this section, I present the Markowitz Model and the Index Model, explaining their importance in portfolio theory and our broader objective of unraveling the complex relationship between risk and return.

3.1 The Markowitz Model: Optimal Diversification for Risk-Adjusted Returns

The Markowitz mean-variance model is based on a set of key assumptions that shape both its theoretical foundation and practical application in investment analysis:

1)Rational Investor Behavior: The model assumes that investors behave rationally, seeking to achieve optimal returns while minimizing the risks inherent in their investment decisions.

2)Simplification of Investor Preference: The model simplifies investor behavior by focusing exclusively on return and risk, deliberately excluding external factors such as taxes or transaction costs. This allows for a foundational understanding of portfolio optimization.

3)Divisibility of Assets: A central assumption of the model is that assets can be infinitely divided, enabling the precise allocation of resources across different investment options and enhancing portfolio construction.

4)Mean and Variance as Decision Metrics: The model posits that investors assess their portfolios based on two key statistical measures: the expected mean (average) return and the variance (or standard deviation) of returns. These metrics capture the distribution and central tendency of potential outcomes, guiding investment decisions.

3.2 The Index Model: A Simpler Framework for Systematic Risk Analysis

The Index Model complements the Markowitz Model by providing a simplified yet insightful method for portfolio analysis. It views the entire market as a unified entity, mirroring the collective behavior of investors. By investing in an index fund that tracks market performance, investors can gain broad exposure to the market and potentially mitigate the effects of individual stock volatility. This analysis will explore the mechanics of the Index Model and its practical applications using the dataset.

William Sharpe's single-index model is grounded on two fundamental assumptions that enhance its simplicity and ease of calculation:

1) Risk Decomposition: This assumption divides security risk into systematic and idiosyncratic components. Systematic risk is influenced solely by external factors like market indices, while idiosyncratic risk remains unaffected by these factors.

2) Independent Idiosyncratic Risk: The second assumption asserts that the idiosyncratic risk of one security is independent of the idiosyncratic risks associated with other securities. Instead, the correlation between the returns of two securities is solely due to their shared response to influencing factors.

These assumptions lead to a significant simplification in the single-index model. By isolating the relationship between systematic risk and external factors and decoupling the idiosyncratic risks among securities, the model greatly simplifies the calculation process. This streamlined approach is particularly advantageous when estimating expected returns and evaluating portfolio diversification strategies.

3.3 Additional Constraints

3.3.1 Explanation of Notation:

w_i: Represent the weight of the i^{th} asset in the portfolio. Particularly, w_1 represents the weight of the SPX Index.

3.3.2 Additional Constraints and Their Explanations:

1)This constraint simulates the Regulation T by FINRA, which allows broker-dealers to permit their customers to have positions, with 50% or more of those positions funded by the customer's account equity. The constraint can be expressed as:

$$\sum_{i=1}^{22} | w_i | \leqslant 2$$

2)This constraint is designed to simulate constraints on the weights of the assets, which might be imposed by the client. This limits the weight of each asset in the portfolio to a maximum of 1 (or another specified value) to prevent overly concentrated positions. The constraint is:

$$|w_i| < 1, \forall i$$

3)This is a free problem, to illustrate how the area of permissible portfolios in general and the efficient frontier in particular look like if there is no constraint.

4)This constraint simulates the typical limitations existing in the U.S. mutual fund industry: a U.S. open-ended mutual fund is not allowed to have any short positions:

$$|w_i| \geqslant 0, \forall i$$

5) Lastly, we would like to see if the inclusion of the broad index into our portfolio has positive or negative effect, for that we would like to consider an additional optimization constraint:

$$w_1 = 0$$

4. The Data Analysis for Markowitz Model and the Index Model

4.1 Graphical Results

In empirical analysis, verifying the distributional characteristics of data is crucial to ensure the validity of model assumptions. Given that the normal distribution serves as a core assumption for the Markowitz Model, this study first examines the normality of the data through visualization methods. Specifically, I plotted the distribution of data overlaid with theoretical normal distribution curves.

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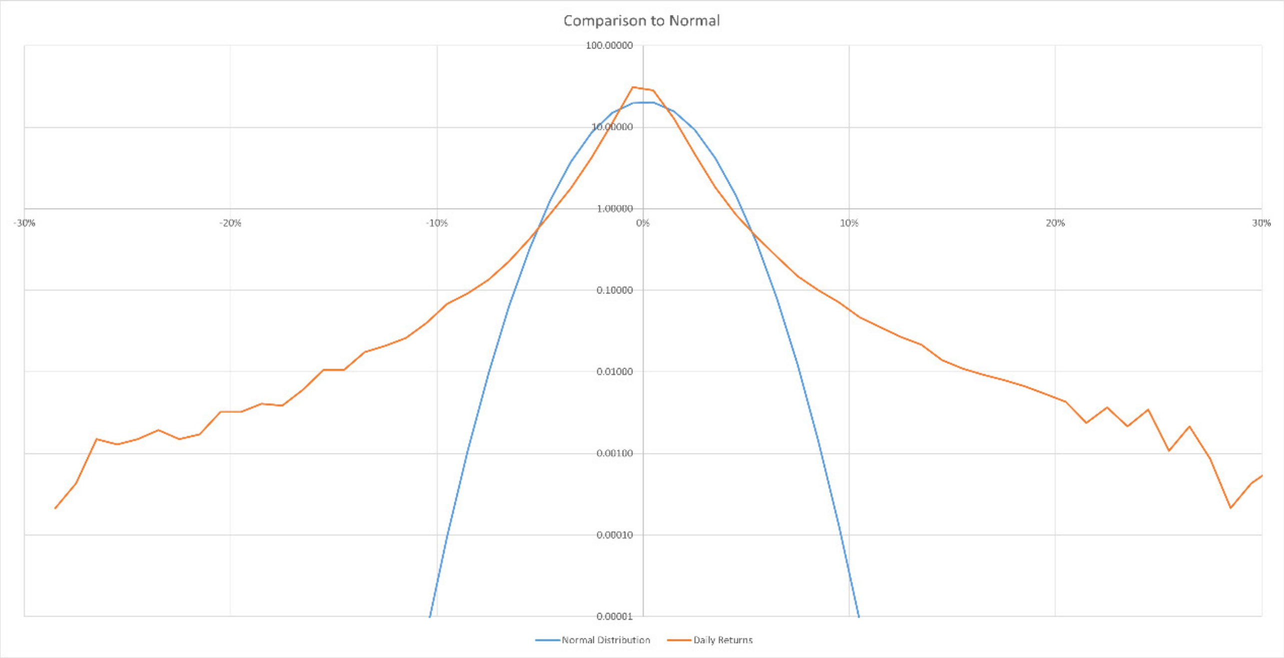

Figure 4.1 Comparison of daily data and normal distribution

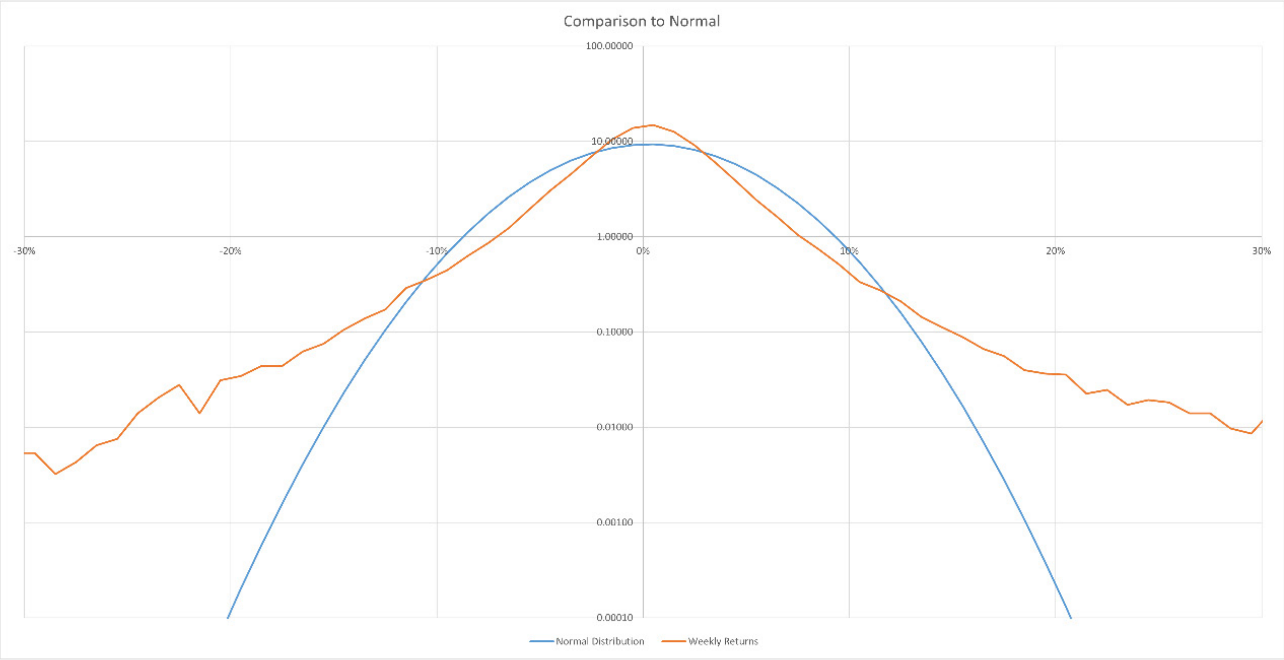

Figure 4.2 Comparison of weekly data and normal distribution

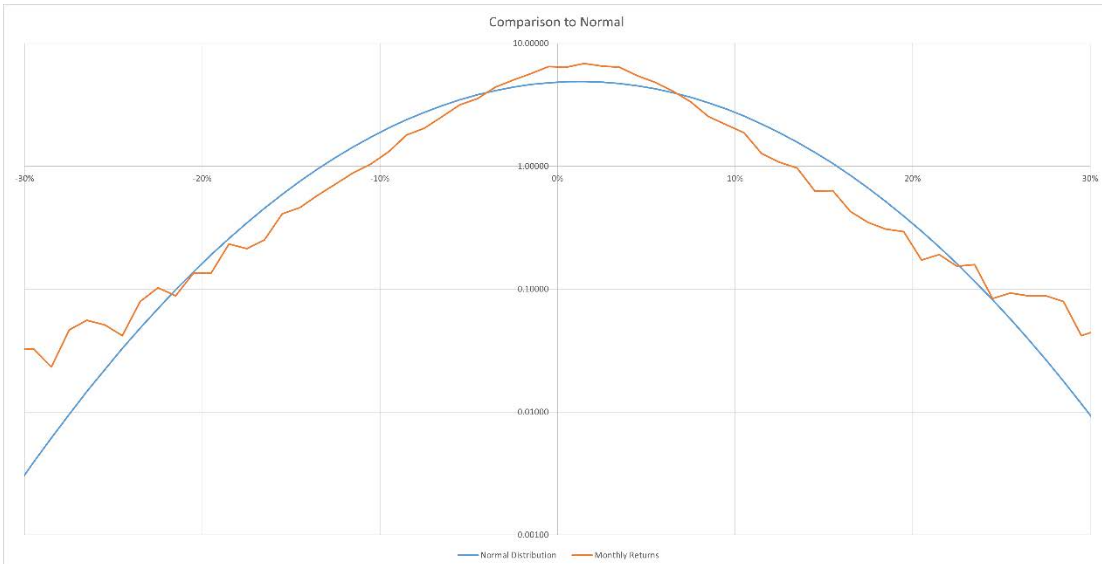

Figure 4.3 Comparison of monthly data and normal distribution

A comparison of Figures 4.1, 4.2 and 4.3 reveals that the distribution of monthly return data exhibits greater uniformity and more closely aligns with the normal probability density function curve.

4.2 Correlation Analysis

Table 4.4 The correlations of the total returns between each stock

	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
SPX	100%	66%	67%	60%	59%	76%	80%	70%	65%	66%	66%	68%	62%	57%	71%	58%	56%	55%	55%	55%	69%	69%
JPM	66%	100%	63%	73%	77%	49%	55%	51%	46%	46%	50%	47%	40%	22%	40%	41%	34%	31%	24%	38%	42%	42%
MS	67%	63%	100%	42%	44%	49%	61%	46%	51%	41%	42%	48%	33%	38%	46%	42%	36%	34%	30%	41%	48%	51%
USB	60%	73%	42%	100%	77%	44%	45%	50%	42%	49%	50%	44%	47%	16%	38%	37%	22%	28%	14%	28%	35%	30%
WFC	59%	77%	44%	77%	100%	43%	48%	44%	41%	43%	43%	42%	41%	16%	32%	28%	26%	24%	15%	28%	32%	32%
EMR	76%	49%	49%	44%	43%	100%	71%	74%	64%	64%	61%	58%	52%	39%	51%	48%	35%	49%	41%	40%	60%	58%
HON	80%	55%	61%	45%	48%	71%	100%	67%	64%	57%	55%	59%	46%	38%	57%	51%	37%	43%	36%	45%	58%	62%
CAT	70%	51%	46%	50%	44%	74%	67%	100%	69%	57%	61%	54%	52%	31%	47%	42%	35%	42%	35%	36%	53%	55%
DE	65%	46%	51%	42%	41%	64%	64%	69%	100%	54%	50%	54%	47%	33%	39%	40%	32%	37%	34%	37%	49%	53%
MMM	66%	46%	41%	49%	43%	64%	57%	57%	54%	100%	57%	50%	51%	36%	45%	42%	28%	38%	31%	34%	54%	50%
FDX	66%	50%	42%	50%	43%	61%	55%	61%	50%	57%	100%	54%	70%	33%	45%	37%	34%	38%	36%	36%	45%	49%
UNP	68%	47%	48%	44%	42%	58%	59%	54%	54%	50%	54%	100%	50%	37%	50%	45%	38%	39%	31%	42%	53%	55%
UPS	62%	40%	33%	47%	41%	52%	46%	52%	47%	51%	70%	50%	100%	28%	40%	31%	39%	33%	34%	36%	41%	50%
AAPL	57%	22%	38%	16%	16%	39%	38%	31%	33%	36%	33%	37%	28%	100%	43%	33%	37%	42%	45%	39%	46%	46%
ACN	71%	40%	46%	38%	32%	51%	57%	47%	39%	45%	45%	50%	40%	43%	100%	53%	38%	38%	37%	37%	46%	51%
IBM	58%	41%	42%	37%	28%	48%	51%	42%	40%	42%	37%	45%	31%	33%	53%	100%	35%	41%	31%	31%	42%	41%
AMD	56%	34%	36%	22%	26%	35%	37%	35%	32%	28%	34%	38%	39%	37%	38%	35%	100%	29%	54%	44%	45%	32%
INTC	55%	31%	34%	28%	24%	49%	43%	42%	37%	38%	38%	39%	33%	42%	38%	41%	29%	100%	39%	28%	56%	39%
NVDA	55%	24%	30%	14%	15%	41%	36%	35%	34%	31%	36%	31%	34%	45%	37%	31%	54%	39%	100%	41%	52%	32%
QCOM	55%	38%	41%	28%	28%	40%	45%	36%	37%	34%	36%	42%	36%	39%	37%	31%	44%	28%	41%	100%	50%	39%
TXN	69%	42%	48%	35%	32%	60%	58%	53%	49%	54%	45%	53%	41%	46%	46%	42%	45%	56%	52%	50%	100%	49%
LIN	69%	42%	51%	30%	32%	58%	62%	55%	53%	50%	49%	55%	50%	46%	51%	41%	32%	39%	32%	39%	49%	100%

From the table we can see that all stocks have a correlation with the SPX index higher than 50%. More specifically, unlike stocks in other sectors, most technology stocks (AAPL, IBM, AMD, INTC, NVDA and QCOM) exhibit correlations with the SPX below 60%, reflecting that the technology sector is less affected by broader market movements compared to other industries. Meanwhile, the Financial and Industrial sectors exhibit stronger internal correlations, with most stock correlations within these sectors exceeding 60%. For example, JPM has correlations 63%, 73% and 77% with respect to MS, USB and WFC, respectively. This contrasts with their lower correlations with stocks in other sectors, reflecting the aligned cyclical patterns of Financial and Industrial sectors. Concurrently, Technology sector stocks demonstrate notably weaker connections with other sectors, showing correlations predominantly below 50% - many even ranging between 20% to 30%. This pattern highlights the relatively independent performance characteristics of technology stocks compared to other market segments.

4.3 Regression Analysis for the Index Model

The Index Model is typically expressed as:

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i$$

Where:

· R_i is the return on asset i.

· R_m is the return on the market index (SPX).

· α_i is the alpha of asset i, representing the asset's excess return (the portion of the return not explained by the market).

· β_i is the beta of asset i, measuring the sensitivity of the asset's return to the market return. It reflects the degree to

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which the asset's return moves in relation to the market's return.

· ϵ_i is the residual term for asset i, capturing the portion of the return that is unrelated to market movements (i.e., idiosyncratic risk). It is assumed to be random.

Table 4.5 The regression result (1)

	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP
Annualized Average Return	9.535%	13.066%	10.458%	7.096%	8.865%	10.257%	11.805%	17.271%	16.996%	6.969%	8.771%	17.168%
Annualized Standard Deviation	14.952%	26.927%	33.102%	25.023%	29.766%	24.790%	21.856%	31.649%	28.463%	21.548%	28.885%	23.531%
β With Respect to SPX	100.000%	119.757%	148.185%	99.651%	116.930%	126.075%	117.293%	147.320%	122.989%	94.424%	126.956%	107.492%
α With Respect to SPX	0.000%	1.648%	-3.672%	-2.406%	-2.284%	-1.764%	0.621%	3.224%	5.270%	-2.034%	-3.334%	6.919%
Residual Standard Deviation	0.000%	20.110%	24.592%	20.103%	24.091%	16.099%	13.043%	22.725%	21.724%	16.278%	21.771%	17.186%

Table 4.6 The regression result (2)

	SPX	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
Annualized Average Return	9.535%	6.650%	33.900%	15.113%	8.448%	27.732%	5.392%	44.640%	12.974%	14.528%	14.023%
Annualized Standard Deviation	14.952%	23.235%	32.458%	21.861%	20.983%	58.909%	27.976%	47.981%	32.727%	24.095%	18.397%
β With Respect to SPX	100.000%	96.896%	123.828%	104.242%	80.981%	220.723%	102.493%	175.863%	120.789%	111.827%	84.660%
α With Respect to SPX	0.000%	-2.588%	22.094%	5.173%	0.727%	6.687%	-4.381%	27.871%	1.457%	3.865%	5.951%
Residual Standard Deviation	0.000%	18.165%	26.659%	15.329%	17.137%	48.796%	23.405%	40.134%	27.292%	17.348%	13.349%

Among the stocks analyzed, NVDA stands out with the highest annualized return of 44.64% and a positive alpha of 27.87%, indicating significant outperformance relative to the market. AAPL also shows strong performance with a 33.90% return and a positive alpha of 22.09%. However, AMD, despite its high return of 27.73%, has a negative alpha (-4.38%), suggesting it underperformed relative to market expectations. JPM offers a moderate return of 13.07% and a positive alpha of 1.65%, while WFC underperforms with the lowest return (7.10%) and a negative alpha (-2.28%). The analysis highlights that tech stocks tend to offer higher returns but come with greater volatility, while traditional financial stocks show more stability but with lower performance.

5. Markowitz Model (MM) and Index Model (IM) Comparisons

In this part I will compare the capital allocation line (CAL) of MM and IM under different constraints, which are listed in part 3.3.2. For each constraint I calculated the minimum variance and maximum Sharpe ratio point for MM and IM.

5.1 The constraint 1) in part 3.3.2, which is:

$$\sum_{i=1}^{22} |w_i| \leq 2$$

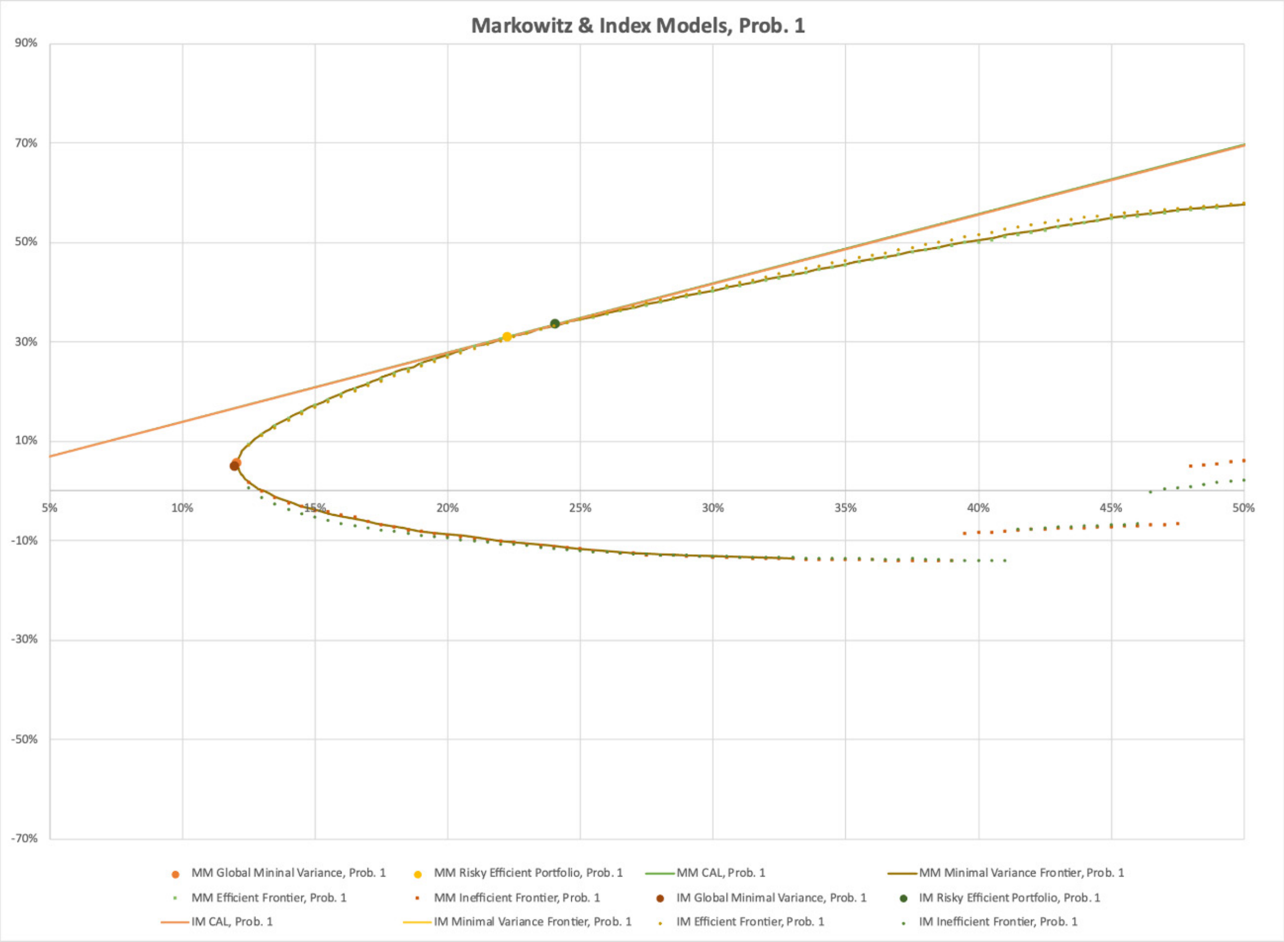
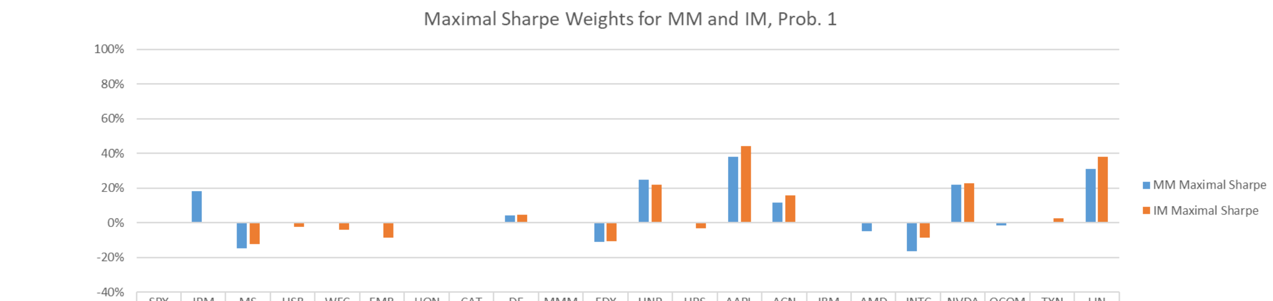


Figure 5.1 The efficient frontiers and CAL (Constraint 1)



	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
MM Maximal Sharpe	0.00%	18.32%	-14.83%	0.09%	0.04%	-0.33%	-0.02%	0.06%	4.20%	-0.07%	-11.27%	24.59%	-0.05%	38.09%	11.63%	0.06%	-4.77%	-16.60%	21.70%	-1.76%	0.03%	30.87%
IM Maximal Sharpe	-0.01%	0.00%	-12.36%	-2.36%	-3.93%	-8.48%	-0.03%	0.03%	4.64%	-0.42%	-10.54%	21.78%	-3.11%	44.05%	15.73%	0.04%	0.02%	-8.45%	22.78%	0.01%	2.51%	38.11%

Figure 5.2 The efficient Sharpe ratio for MM and IM (Constraint 1)

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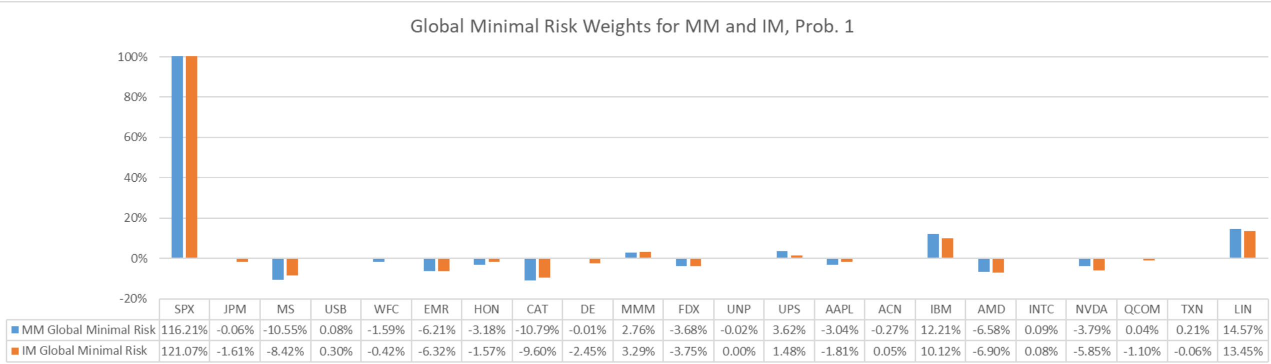


Figure 5.3 The minimal risk for MM and IM (Constraint

5.2 The constraint 2) in part 3.3.2, which is:

$$|w_i| < 1, \forall i$$

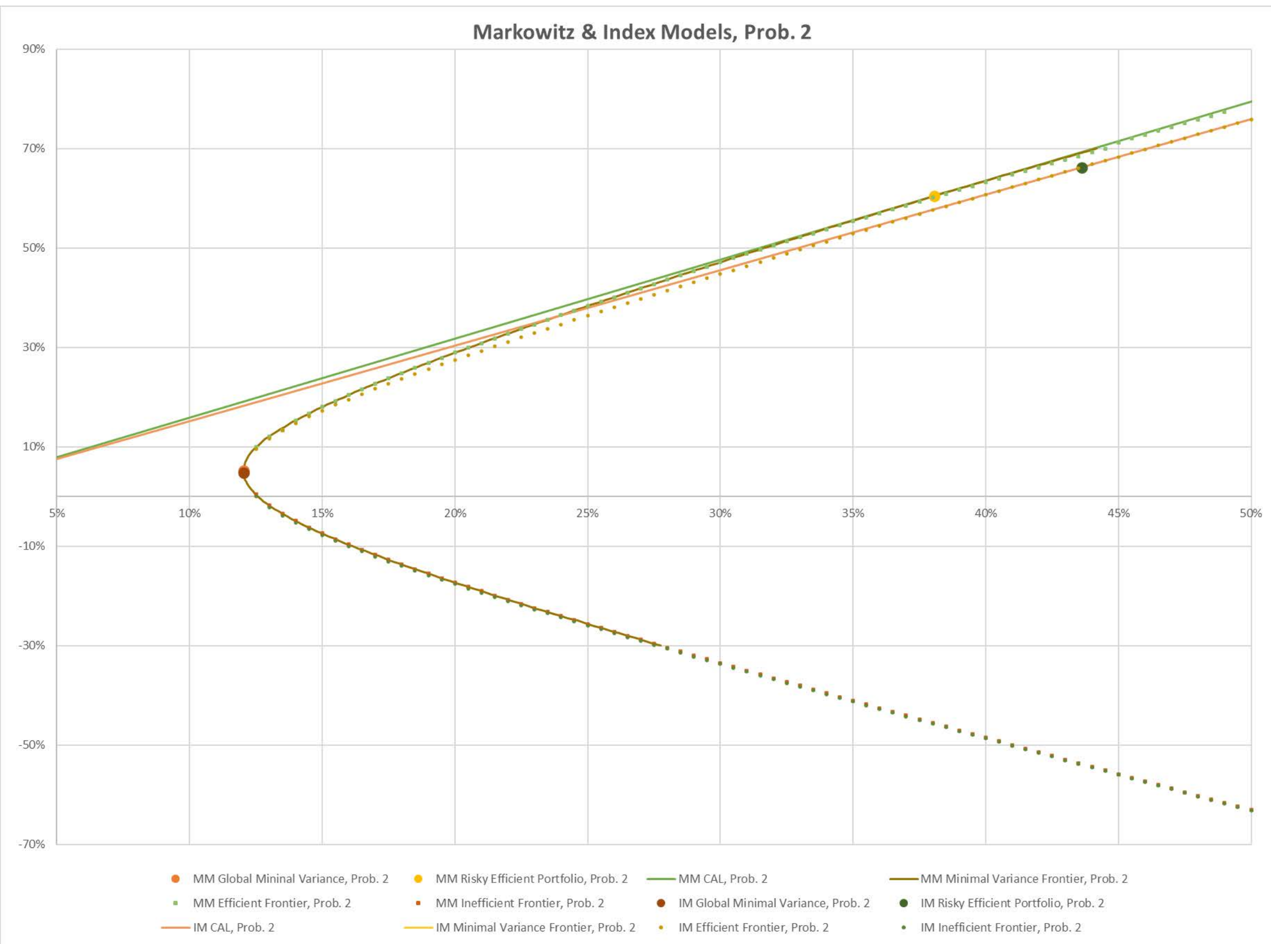
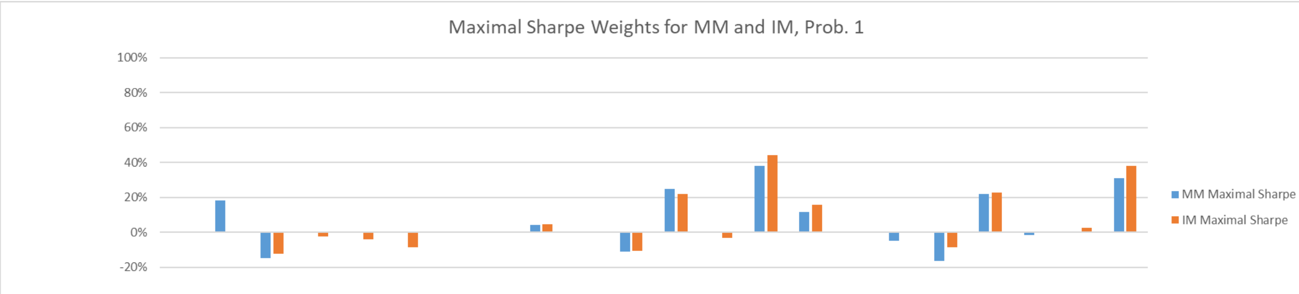
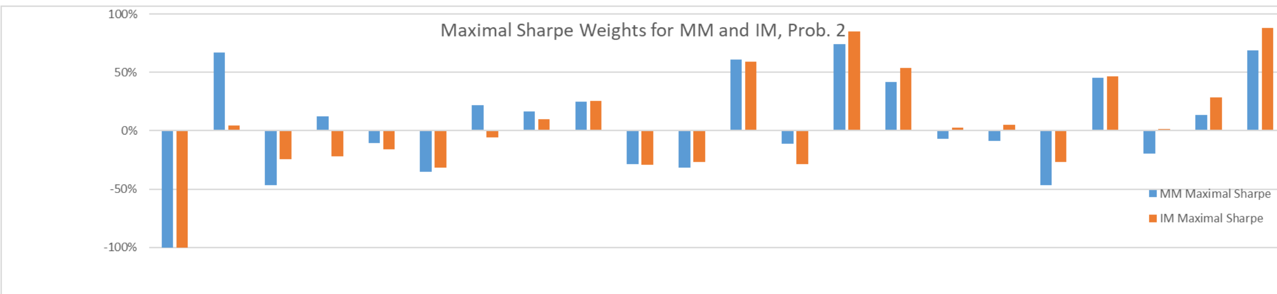


Figure 5.4 The efficient frontiers and CAL (Constraint 2)



	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
MM Maximal Sharpe	0.00%	18.32%	-14.83%	0.09%	0.04%	-0.33%	-0.02%	0.06%	4.20%	-0.07%	-11.27%	24.59%	-0.05%	38.09%	11.63%	0.06%	-4.77%	-16.60%	21.70%	-1.76%	0.03%	30.87%
IM Maximal Sharpe	-0.01%	0.00%	-12.36%	-2.36%	-3.93%	-8.48%	-0.03%	0.03%	4.64%	-0.42%	-10.54%	21.78%	-3.11%	44.05%	15.73%	0.04%	0.02%	-8.45%	22.78%	0.01%	2.51%	38.11%

Figure 5.5 The efficient Sharpe ratio for MM and IM (Constraint 2)



	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
MM Maximal Sharpe	-100.00%	66.82%	-46.86%	12.09%	-10.81%	-35.19%	21.77%	16.74%	24.72%	-28.65%	-31.84%	61.09%	-11.37%	74.53%	41.50%	-6.97%	-8.62%	-46.68%	45.21%	-19.78%	13.55%	68.76%
IM Maximal Sharpe	-100.00%	4.56%	-24.11%	-22.14%	-16.05%	-31.60%	-5.85%	10.19%	25.71%	-28.93%	-26.87%	59.28%	-28.36%	85.01%	53.74%	2.69%	5.12%	-26.90%	46.48%	1.67%	28.45%	87.91%

Figure 5.6 The minimal risk for MM and IM (Constraint 2)

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5.3 The constraint 3) in part 3.3.2, which is a free constraint.

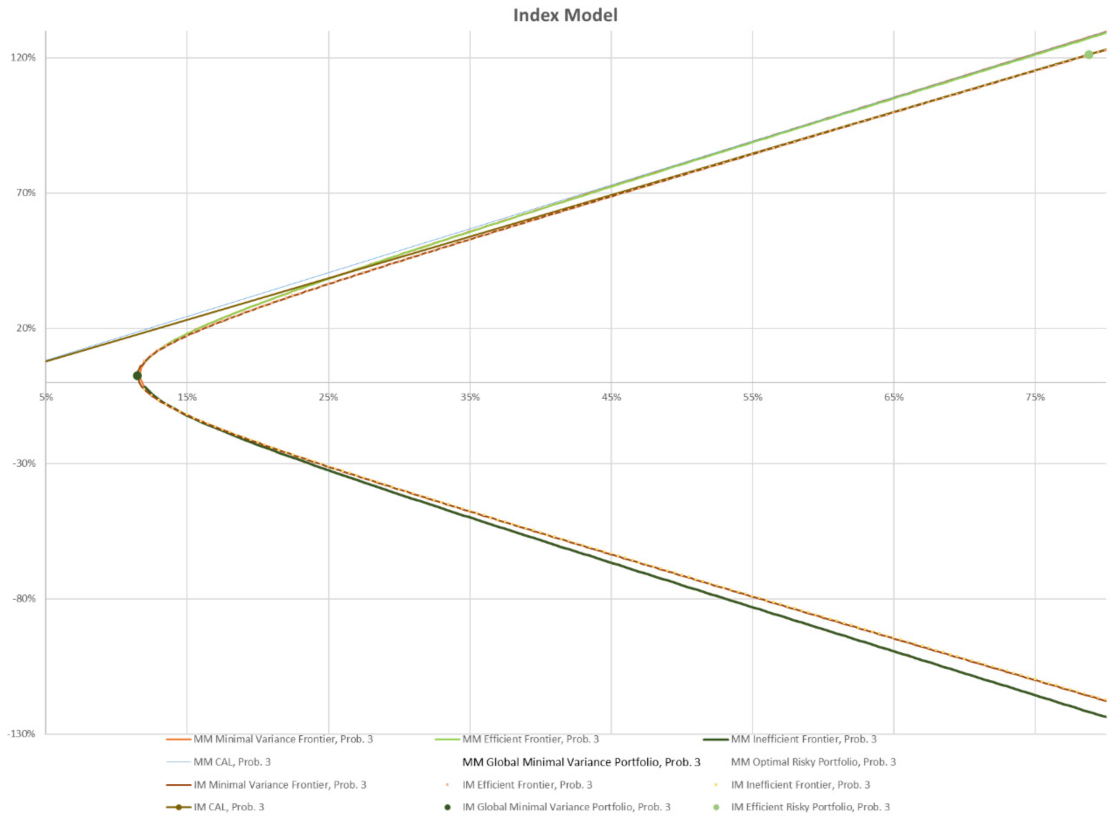
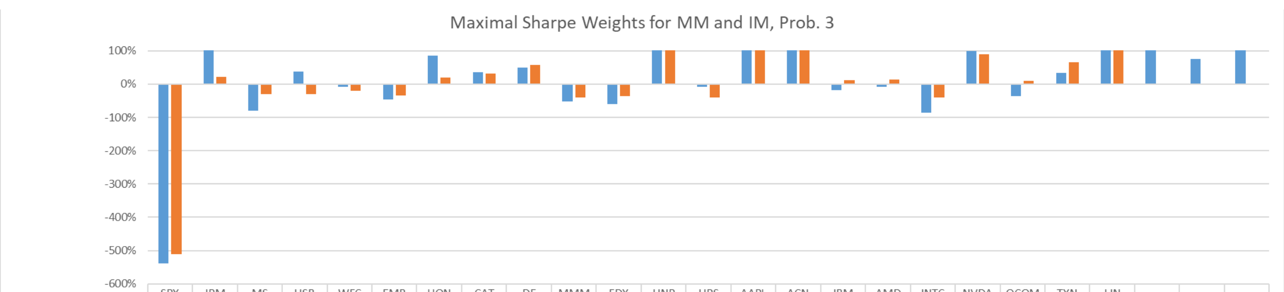
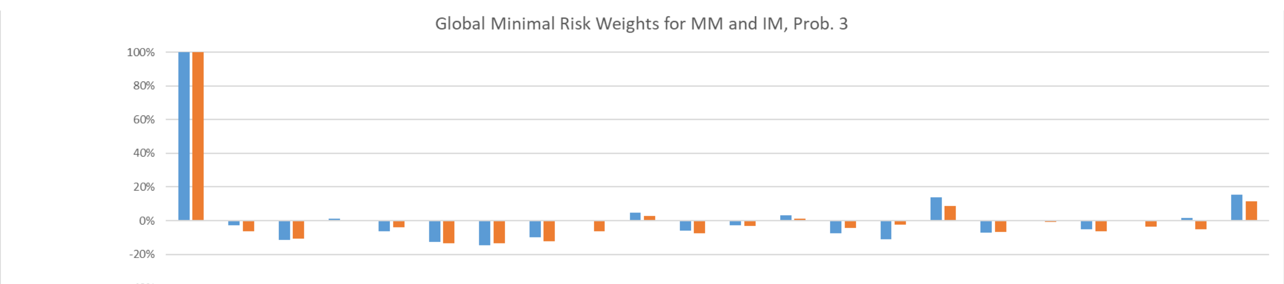


Figure 5.7 The efficient frontiers and CAL (Constraint 3)



	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN			
MM Maximal Sharpe	-538.79	141.77%	-79.10%	37.71%	-7.25%	-45.93%	84.54%	36.06%	48.67%	-52.22%	-60.65%	128.12%	-8.40%	164.31%	118.55%	-17.39%	-7.93%	-85.72%	98.85%	-35.98%	33.99%	146.80%	123.47%	76.21%	1.620
IM Maximal Sharpe	-510.43	20.88%	-31.11%	-30.51%	-20.17%	-34.88%	18.71%	32.00%	57.22%	-39.35%	-36.05%	120.04%	-40.20%	159.31%	112.82%	12.68%	14.39%	-40.98%	88.67%	10.02%	65.81%	171.12%			

Figure 5.8 The efficient Sharpe ratio for MM and IM (Constraint 3)



	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
MM Global Minimal Risk	157.13%	-2.83%	-11.29%	1.05%	-6.49%	-12.64%	-14.53%	-10.05%	0.52%	4.76%	-6.08%	-2.58%	3.24%	-7.65%	-11.11%	13.74%	-6.99%	-0.33%	-5.20%	0.16%	1.79%	15.38%
IM Global Minimal Risk	181.87%	-6.44%	-10.50%	0.11%	-3.84%	-13.25%	-13.39%	-12.07%	-6.42%	2.77%	-7.49%	-3.34%	1.24%	-4.42%	-2.38%	8.53%	-6.68%	-0.60%	-6.20%	-3.68%	-5.18%	11.34%

Figure 5.9 The minimal risk for MM and IM (Constraint 3)

5.4 The constraint 4) in part 3.3.2, which is:

$$\left|w_i\right| \geq 0, \forall i$$

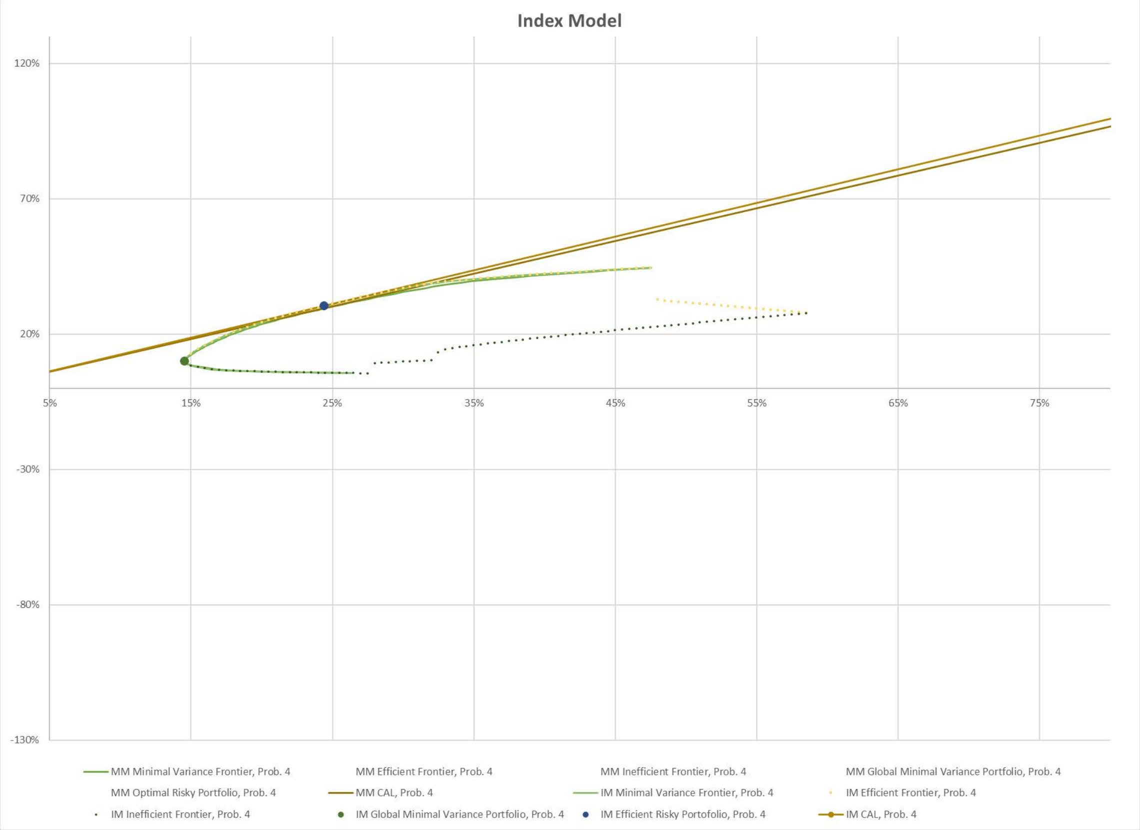
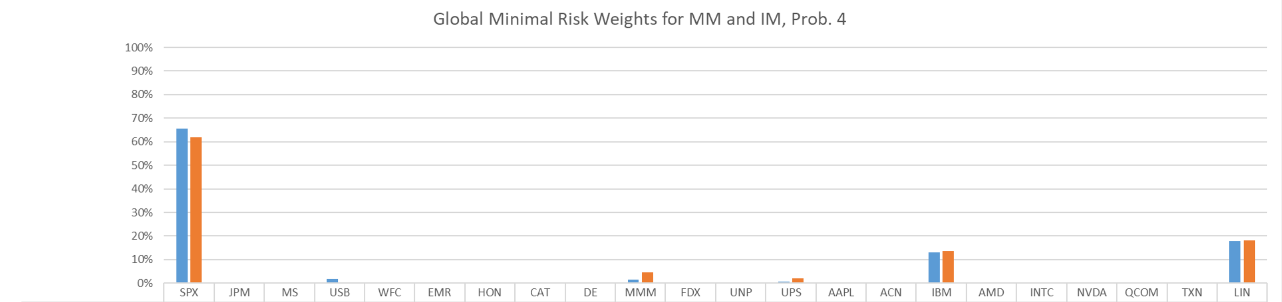


Figure 5.10 The efficient frontiers and CAL (Constraint 4)



	SPX	JPM	MS	USB	WFC	EMR	HON	CAT	DE	MMM	FDX	UNP	UPS	AAPL	ACN	IBM	AMD	INTC	NVDA	QCOM	TXN	LIN
MM Global Minimal Risk	65.57%	0.00%	0.00%	1.66%	0.01%	0.00%	0.00%	0.00%	0.00%	1.53%	0.00%	0.00%	0.54%	0.00%	0.00%	12.94%	0.00%	0.00%	0.00%	0.00%	0.00%	17.76%
IM Global Minimal Risk	61.77%	0.00%	0.00%	0.13%	0.00%	0.00%	0.00%	0.00%	0.00%	4.44%	0.00%	0.00%	1.98%	0.00%	0.00%	13.66%	0.00%	0.00%	0.00%	0.00%	0.00%	18.03%

Figure 5.11 The efficient Sharpe ratio for MM and IM (Constraint 4)

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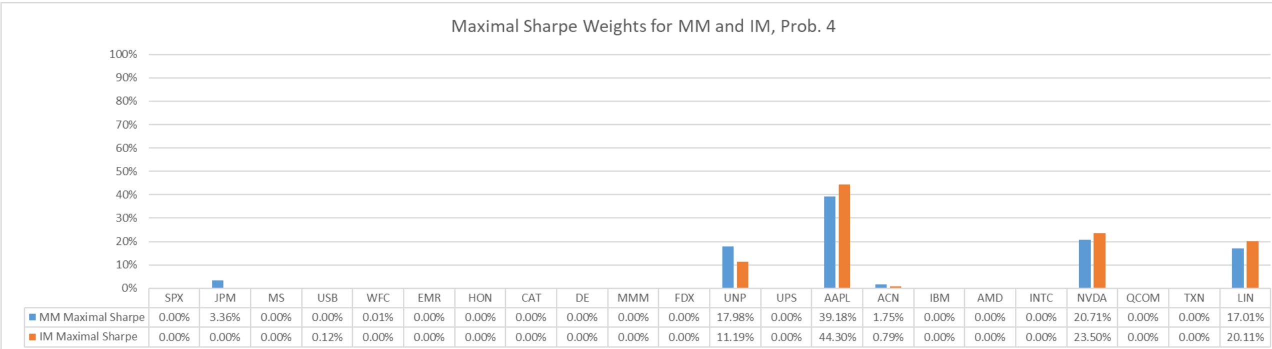

Figure 5.12 The minimal risk for MM and IM (Constraint 4)

5.5 The constraint 5) in part 3.3.2, which is: $w_1 = 0$

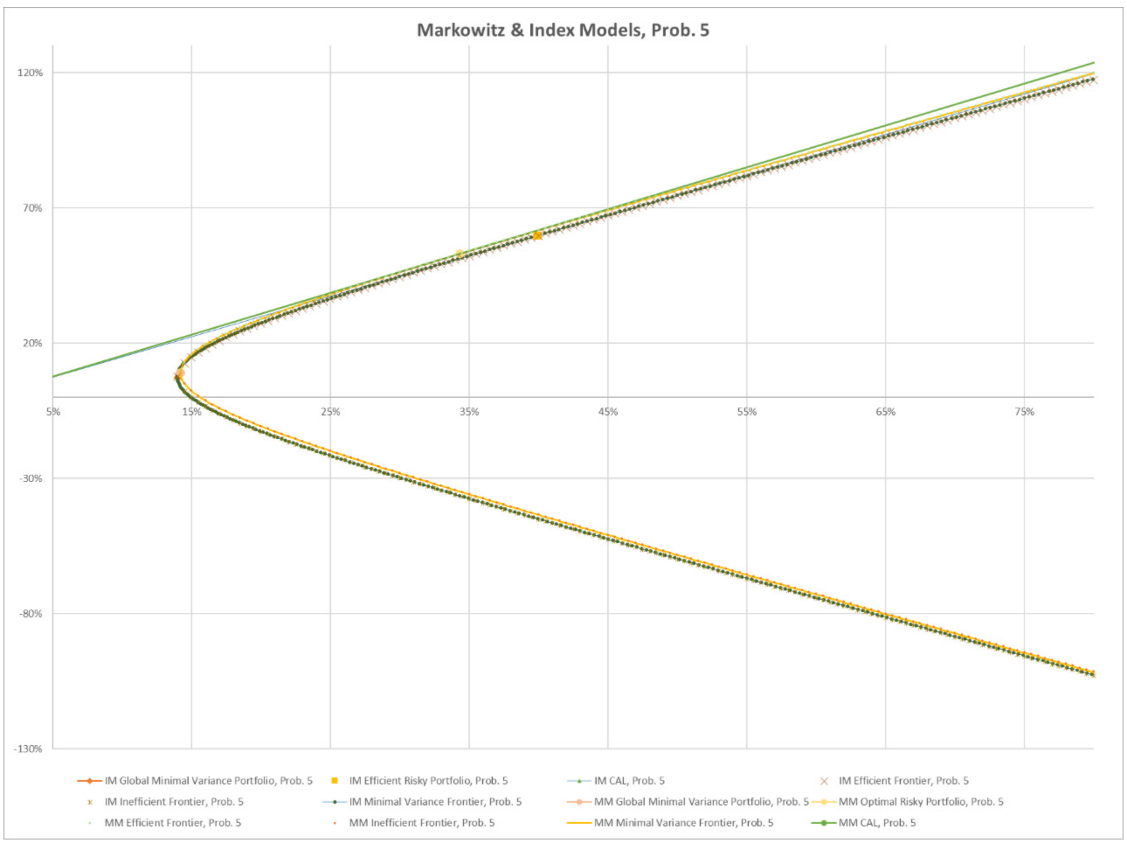

Figure 5.13 The efficient frontiers and CAL (Constraint 5)

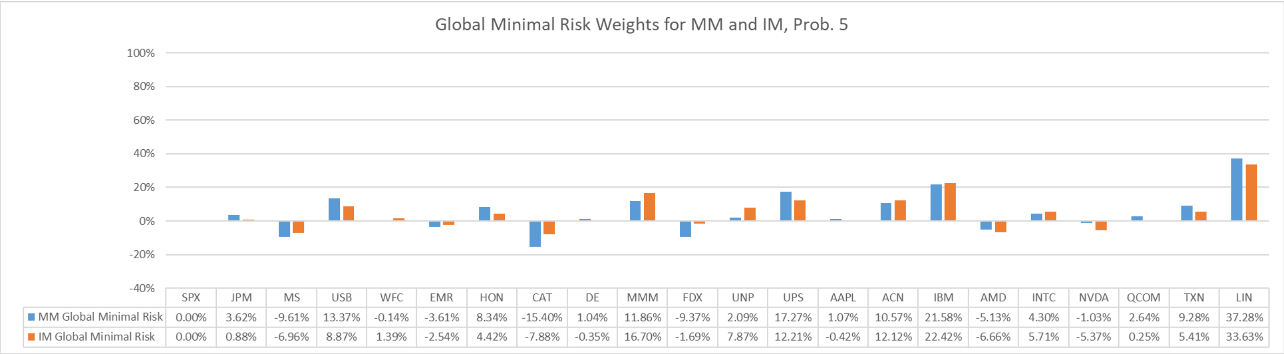

Figure 5.14 The efficient Sharpe ratio for MM and IM (Constraint 5)

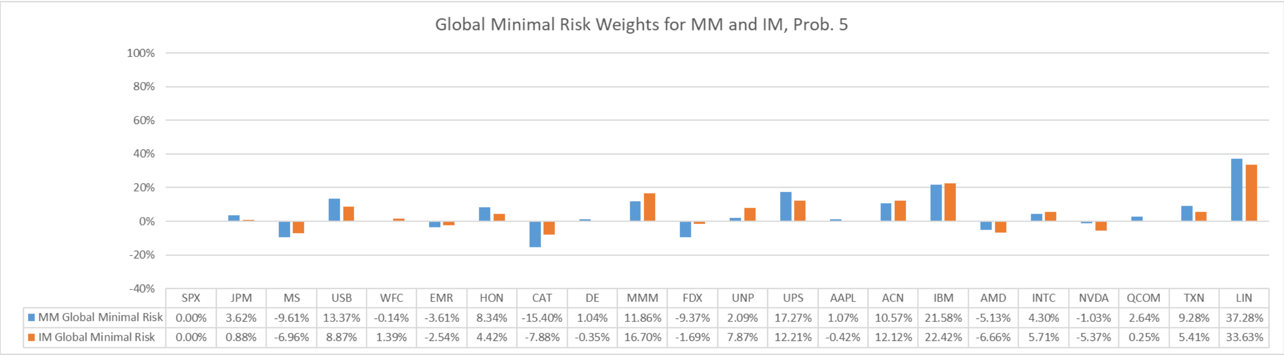

Figure 5.15 The minimal risk for MM and IM (Constraint 5)

5.6 The Numerical Comparison

To further assess the effectiveness of the Index Model, numerical comparisons between the maximum Sharpe ratio points and the minimum variance points are also necessary.

	Return for Minimal Risk		Efficient Sharpe Ratio	
	MM	IM	MM	IM
Constraint 1	5.59%	4.99%	1.393	1.388
Constraint 2	5.16%	4.72%	1.589	1.519
Constraint 3	2.89%	2.57%	1.620	1.537
Constraint 4	10.10%	10.02%	1.208	1.245
Constraint 5	9.10%	7.34%	1.546	1.495

Figure 5.16 Numerical results of MM and IM (1)

	Sharpe Ratio Under Minimal Risk		Risk Under Efficient Sharpe Ratio	
	MM	IM	MM	IM
Constraint 1	0.463	0.416	22.27%	24.08%
Constraint 2	0.429	0.392	38.06%	43.61%
Constraint 3	0.248	0.224	76.21%	78.78%
Constraint 4	0.694	0.690	23.75%	24.40%
Constraint 5	0.640	0.524	34.31%	39.90%

Figure 5.17 Numerical results of MM and IM (2)

Now we analyze the numerical results of different constraints.

1) Constraint 1: The Markowitz model outperforms the Index model in terms of risk-adjusted returns under con-

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straint 1. The Markowitz model's minimum risk portfolio offers slightly higher returns (5.59% vs. 4.99%) and a better Sharpe ratio (0.463 vs. 0.416). Similarly, the Markowitz model's maximum Sharpe ratio portfolio shows a slightly superior Sharpe ratio (1.393 vs. 1.388) with lower risk (22.27% vs. 24.08%) compared to the Index model. Overall, the Markowitz model provides a more efficient risk-return trade-off in both portfolios under constraint 1.

2) Constraint 2: For the minimum risk portfolio, the Markowitz model yields a return of 5.16% with a standard deviation of 12.03% and a Sharpe ratio of 0.429, while the Index model provides a return of 4.72% with the same standard deviation of 12.03% and a slightly lower Sharpe ratio of 0.392. Thus, the Markowitz model provides slightly better risk-adjusted returns, consistent with the previous results. In the case of the maximum Sharpe ratio portfolio, the Markowitz model achieves a Sharpe ratio of 1.589 with a standard deviation of 38.06%, while the Index model shows a slightly lower Sharpe ratio of 1.519 with a higher standard deviation of 43.61%. The Markowitz model demonstrates a better risk-return trade-off with a higher Sharpe ratio.

3) Constraint 3: In the free problem, the Markowitz model continues to outperform the Index model in terms of risk-adjusted returns, particularly in the maximum Sharpe ratio portfolio.

4) Constraint 4: The Markowitz model and Index model show similar results in both minimum risk and maximum Sharpe ratio portfolios. For the minimum risk portfolio, the Markowitz model has a slightly higher Sharpe ratio (0.694 vs. 0.690). In the maximum Sharpe ratio portfolio, the Index model shows a marginally higher standard deviation, resulting in a slightly higher Sharpe ratio (1.245 vs. 1.208). Overall, both models exhibit comparable performance, with the Markowitz model showing a slight edge in risk-adjusted return for the minimum risk portfolio, while the Index model outperforms in terms of return for the maximum Sharpe ratio portfolio. This is the only constraint where the efficient Sharpe ratio performance of MM is worse than that of IM. The reason might be that the Markowitz model fail to fully utilize the low correlation between assets due to the inability to short sell, thereby affecting the final portfolio performance.

5) Constraint 5: The Markowitz model outperforms the Index model in terms of risk-adjusted return for the minimum risk portfolio, with a Sharpe ratio of 0.640 compared to 0.524. For the maximum Sharpe ratio portfolio, the Markowitz model delivers a higher Sharpe ratio of 1.546. Overall, the Markowitz model demonstrates better risk-adjusted performance, indicating it is more efficient in managing risk relative to returns.

6. Conclusion

This paper mainly discussed the outcomes of portfolio optimization under MM and IM, particularly under real-world conditions where the models' theoretical assumptions may not hold. The analytical approach begins with a critical examination of the underlying assumptions. To ensure that the data roughly follows a normal distribution, this paper ultimately chooses to use monthly data for the calculations.

Next, I calculated the correlation coefficients between the index and each stock. These coefficients reflect the relationships between each security, the other stocks, and the portfolio as a whole. The findings show that all the selected stocks exhibit positive correlations with one another and with the index. Additionally, the strength of these correlations may vary across different sectors.

Then, five different additional constraints are applied to the weight of each stock in the asset portfolio. From the plotted frontier figures, we can observe that the efficient frontiers, inefficient frontiers, and minimal variance frontiers of MM and IM are nearly identical.

Finally, the numerical results demonstrate that, under most constraints, MM outperforms IM in optimization when calculating both the minimal risk and the efficient Sharpe ratio. This may be due to the fact that this paper uses only 21 stocks for the calculations. Therefore, MM's method of calculating the correlations between each stock and optimizing accordingly is more precise, making it more effective than IM's broader optimization approach. The only constraint where MM underperforms IM is the constraint 4, which prohibits short selling. The reason could be that the Markowitz model fails to fully leverage the low correlation between assets due to the restriction on short selling, which in turn impacts the final portfolio performance. Overall, when the number of stocks is small, the MM model demonstrates better portfolio allocation capabilities than the IM model. At the same time, the gap between the IM and MM models is relatively small, and under certain conditions (such as constraint 4), the IM model shows better allocation capabilities. Therefore, both models have practical value in real-world investment problems.

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