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A Study of Black-Scholes-Merton Model

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Abstract:

This study introduces the Option Pricing Theory improved by Black, Scholes, and Merton, which is also known as the Black-Scholes-Merton model, in detail and investigates its derivation process and historical development process. Despite the content above, it also includes its characteristics and mechanism. The newest Option Pricing formula proposed by Black, Scholes, and Merton was derived from mathematical methods such as the stochastic integral equation and Ito theorem. This essay concludes that the Black Scholes Merton model paved the way for the development and innovation of the subject of Financial Mathematics, which was found by the person who first came up with the Option Pricing Theory, which also contributes to the investment field to reduce the risk.

Keywords: Black-Scholes-Merton Model, Option Pricing Theory, Investment.

1. Introduction

There are many uncontrollable factors in financial markets. Investors gather the funds in the financial markets, which leads to the common wants of every investor to realize the maximum benefits of the financial markets. In this high-risk context, the financial model was born [1]. There are some common financial models, such as the cash flow model [2], the leveraged buyout model [3] and so on. In addition, we will talk about the option pricing model. Option Pricing Theory is a theory that involves the best case of buying options and Black Scholes analysis and so on. This theory has many predecessors so the foundation of the Black Scholes Merton model is based on the previous option pricing theory. Initial option pricing theory was proposed by Bachelier in 1900 and improved by Black and Scholes recently, which exhibit low risk and foresight that is important to the development of financial mathematics, pilot wave theory [4], and decision analysis [5]. Bachelier's option pricing theory mainly contributes to the area of investment, especially for the people who are hesitant to decide to buy or sell the options, which can provide the opportunity to gain from the price changes of the target, thereby improving the efficiency of investment. Accordingly, it was usually applied to set prices for options and estimate the price of stock. Moreover, the theory had some intrinsic limitations. For instance, it is not practical enough since the value of stock price can be negative in the first founder Bachelier' s theory [6], which is not realistic. According to Sprenkle [7] and Boness [8], Bachelier's function also has no awareness about the influence of time taken. So after a while, they improve the functions by bringing in the new thoughts that yields follow a normal distribution while stock prices follow a lognormal distribution rather than perversion. In particular, the values in time have been developed with precision, which fundamentally promotes the original formula. Although Sprenkle and Boness's function exhibits an encouraging combination of high accuracy and more comprehensive consideration, the option pricing theory requires considering the effect of the increased rate of stock to fit in the practical situation. However, if this term is added, it cannot be substituted to the formula as it is opposite to a basic rule in Finance: For tradable goods, under free trading conditions, the same commodity should have the same price in different markets due to the existence of Arbitrage. That is, the price of a tradable commodity in different countries measured in the same currency should be consistent without taking into account transaction costs. It is named the Law of One Price [8].

The option growth rate is a promising candidate for improving the option pricing function since it addresses the problem of the previous Sprenkle and Boness's formula this essay discussed above. An Economist Samuelson [7] new formula made some improvements in the root of Sprenkle and Boness's proposes by introducing the distinct risk of options and assets since options are derivatives. Samuelson also made contributions to emphasize the founder of option pricing theory should be Bachelier [9]. But the function he pointed out was similar to Sprenkle and Boness's. As a consequence, it is still inadequate to apply to the realistic stock pricing situation.

In this article, we introduce the Black-Scholes Merton model, which is the newest option pricing theory in financial mathematics from the characteristics and deducing. This up-to-date function was proposed by Black and Scholes in 1973 [10] and Merton also had the same conclusion in the early 1970s coincidently [10,11]. As a result, the new option pricing theory is also called the Black -Scholes model. Also, the Black-Schole model is established by using stochastic differential equations and other mathematical tools, such as the pricing formula of European stock options commonly used today. The formula also became a milestone in options pricing as it is the most practical and useful pricing option pricing formula. The difference with the former is that it has more assumptions. To exemplify, the market has no arbitrage, which means no excess return; Underlying assets do not pay dividends, and there is no tax and transaction cost. Besides, the change of price of underlying assets follows the Brown Motion but it has some restrictions compared to Bachelier's function, such as it always uses the larger value between the zero and the stock price so it is always positive.

2. Mechanism

2.1 . Black-Scholes model assumptions

Related to the Black-Scholes model, there are 4 assumptions below to guarantee the validity of this model:

First of all, the price of the underlying asset is subject to geometric Brownian motion, which can be represented as:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Also, during the option validity period, the risk-free interest rate, the underlying asset return, and the underlying asset return volatility are constant;

Besides, the market is frictionless. There are no taxes and transaction costs;

The last one, the option is a European-style option, i.e., the option is not exercisable until expiration.

2.2 . Self-financing Portfolio

2.2.1 . Evolution of the Portfolio Value

Let the value of the investor's portfolio be $X(t)$ at each moment, and the portfolio is invested in a money market account paying a constant interest r and the stock market, and the stock price process obeys geometric Brownian motion.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Assuming that the share of the stock held by the investor at time t is $\Delta(t)$, position $\Delta(t)$ can be random, but must be compatible with the domain flow corresponding to the Brownian motion $W(t), t \geqslant 0$. The remainder of the portfolio $X(t) - \Delta(t)S(t)$ is invested in money market inflation account [12].

The change in the value of an investor's portfolio $dX(t)$ at time t caused by two factors: the capital appreciation of the stock position $\Delta(t)dS(t)$ and the interest earned on the closed cash position $r(X(t) - \Delta(t)S(t))dt$.

$$dX(t) = rX(t)dt + \Delta(t)(\mu - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$

Consider the change in the value of the discounted portfolio $e^{-rt}X(t)$, according to the Ito-Doeblin formula of discounted portfolio prices, the differential is:

$$d(e^{-rt}X(t)) = \Delta(t)(\mu - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t)$$

[13]

2.2.2 . Evolution of Option Values

Consider a European call option that pays $(S(T) - K)^+$ at moment T. Under the knockdown price is some non-negative constant. Black, Scholes, and Merton argue that the value of a call option at any moment depends on the number of hours remaining to the expiration date and the stock price at that moment. If the stock price $S(t) = x$ at moment t, use $c(t, x)$ to denote the value of the call option at the moment.

At the initial moment, the future stock price $S(t)$, and hence the future option value $c(t, S(t))$, is not known, so the objective is to determine the function $c(t, x)$. First, compute the differential of $c(t, S(t))$, and according to the Itô-Doeblin formula, there is:

$$dc(t, S(t)) = [c_t(t, S(t)) + \mu S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt + \sigma S(t)c_x(t, S(t))dW(t)$$

Then calculate the differential of the discounted option value $e^{-rt}c(t, S(t))$, according to the Itô-Doeblin formula is:

$$d(e^{-rt}c(t, S(t)) = e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \mu S(t) c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt + e^{-rt}\sigma S(t)c_x(t, S(t))dW(t)$$

2.2.3 . Evolutionary Equivalence

(the option short) The hedge portfolio invests in stock and money market account with an initial capital $X(0)$ such that at each instant of time $t \in [0, T]$, the portfolio value $X(t)$ is the same as $c(t, S(t))$. This is equivalent for all $t, e^{-rt}c(t, S(t))$. One way to ensure this equality is to guarantee that $d(e^{-rt}X(t)) = d(e^{-rt}c(t, S(t)))$, $\forall t \in [0, T)$ and that $X(0) = c(0, S(0))$.

Comparing the formulas above shows that the above equations hold if and only if

$\Delta(t)(\mu - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) = [-rc(t,S(t)) + \mu S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))]dt + \sigma S(t)c_x(t,S(t))dW(t)$

First, make all terms of $dW(t)$ equal, thus obtaining

$$\Delta(t) = c_x(t,S(t)), \forall t \in [0,T)$$

Then make all terms of dt equal, can get:

$(\mu - r)S(t)c_x(t,S(t)) = -rc(t,S(t)) + c_t(t,S(t)) + \mu S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)), \forall t \in [0,T)$

Simplify it:

$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)), \forall t \in [0,T)$

Therefore, it is necessary to find the continuous function $c(t,x)$ that satisfies the Black-Scholes-Merton partial differential equation

$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x), \forall t \in [0,T), x \geqslant 0$

And the final value condition $c(T,x) = (x-K)^+$ [14]

2.2.4 . Black-Scholes Equation

The differential equation of vanilla call option under the Black-Scholes model is

$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x), \forall t \in [0,T), x \geqslant 0$

Final value condition $c(T,x) = (x-K)^+$.

2.3 . Risk-neutral Pricing

The measure transformation method is used to solve the option price, and the discounted stock price is martingale under the new measure, which is called wind Risk-neutral measure. Without loss of generality, consider the stock price process, $dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$, the stock price is generalized geometric Brownian motion:

$S(t) = S(0)e^{\int_0^t \sigma(s)dW(s) + \int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)ds}$

The interest rate process $r(t)$, which defines the discount process:

$$D(t) = e^{-\int_0^t r(s)ds},\ dD(t) = -r(t)D(t)dt$$

The process of discounting the stock price is

$$D(t)S(t) = S(0)e^{\int_0^t \sigma(s)dW(s) + \int_0^t \mu(s) - r(s) - \frac{1}{2}\sigma^2(s)ds}$$

From the Ito-Doeblin Formula, the differential is equal to

$dD(t)S(t) = (\mu(t) - r(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) = \sigma(t)D(t)S(t)(\theta(t)dt + dW(t))$

Where $\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$, defined as the market price of risk.

Using the Girsanov Theorem, combined with $\theta(t)$ measure transformation, can be obtained

$dD(t)S(t) = \sigma(t)D(t)S(t)dW(t)$

The discounted stock price under this new measure is martingale, which we call the risk-neutral measure.

The dynamic process of stock price under the risk-neutral measure can be obtained by replacing

$dW(t) = -\theta(t)dt + dW(t)$,

$dS(t) = r(t)S(t)dt + \sigma S(t)dW(t)$

It can be seen that the average return of stock price under the risk-neutral measure is the risk-free return rate $r(t)$.

At this point, look at the value of the portfolio process, as seen above

$dX(t) = r(t)X(t)dt + \Delta(t)\sigma(t)S(t)(\theta(t)dt + dW(t)) = r(t)X(t)dt + \Delta(t)\sigma(t)S(t)dW(t)$

The dynamic process of discount portfolio is

$dD(t)X(t) = \Delta(t)\sigma(t)D(t)S(t)dW(t)$

Obviously the discounted portfolio value is martingale,

$D(t)X(t) = \mathbb{E}[D(T)X(T) \mid \mathcal{F}(t)] \Rightarrow X(t) = \mathbb{E}[e^{-\int_t^T r(s)ds} X(T) \mid \mathcal{F}(t)]$

It is clear from the above that the option returns can be replicated by constructing a portfolio of assets to determine the option price based on the no-arbitrage principle. The price of this portfolio can be obtained by calculating the conditional expectation in the above equation.

The risk-neutral measure is equivalent to the actual probability measure in real life, i.e., the risk-neutral measure is equal to the event that is also 0 in the actual probability measure, and an event that is greater than 0 in the risk-neutral measure is also greater than 0 in the actual probability measure.

2.4 . Option Pricing under Black-Scholes Model

2.4.1 . European Options

The risk measure method is used to solve the European option price formula under the Black-Scholes model, and the dynamic process of stock price under the risk-neutral measure is

$dS(t) = rS(t)dt + \sigma S(t)dW(t)$,

Further is

$S(t) = S(0)e^{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)} = S(0)e^{(r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}$, $Z \sim \mathcal{N}(0,1)$

Under the risk-neutral measure, the discounted asset price is a martingale, so there is

$C(0,S(0))=\mathbb{E}\left[e^{-rT}(S(T)-K)^{+}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\infty}e^{-rT}(S(0)$

$e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}-K)^{+}e^{-\frac{1}{2}Z^2}dZ$

[15]

The integrand function

$(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}-K)^{+}$

Take positive if and only if

$$Z<d_2=\frac{ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

And then we can get it by integrating

$C(0,S(0))=S(0)N(d_1)-Ke^{-rT}N(d_2)$,

Wherein, $d_1=\frac{ln\frac{S(0)}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2=d_1-\sigma\sqrt{T},N(\cdot)$ is the cumulative distribution of the normal distribution.

At the same time, according to the definition, there are put-call parity formulas and put option prices

$C(0,S(0))-P(0,S(0))=S(0)-Ke^{-rT}$

$P(0,S(0))=S(0)(1-N(d_1))\_Ke^{-rT}(1-N(d_2))$

2.4.2 American Options

In contrast to European options, American options can be executed at any point before the expiration date. Investors can choose to trade ahead of time due to their self nature, so the first question to consider is: under what circumstances are American options executed.

2.4.2.1 Optimal Execution Boundaries for American Options

From a mathematical point of view, American option pricing is a problem of solving a parabolic partial differential equation with a free boundary. One of the free boundaries is a boundary to be found $S*(t)$, called the optimal execution boundary, which separates the region $[0,+\infty)\times[0,T]$ into two neighboring regions, one is continuing to hold region Σ_1, and the other one is the termination region Σ_2, take an American put option as an example:

in $\Sigma_1=[S*(t),\infty)\times[0,T],V(S,t)>max(K-S,0)$;

in $\Sigma_2=[0,S*(t)]\times[0,T],V(S,t)=max(K-S,0)$.

2.4.2.2 Pricing Equation of American Options

After determining the best implementation boundary, the method and process of establishing the American option pricing model is similar to that of the European option pricing model. Black-Scholes option pricing equation with boundary conditions and final value conditions can be derived. Taking the American put option as an example, the pricing equation is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0,(S,t)\in\Sigma_1$$

The following conditions must also be met:

Final value conditions

$V(S,T)=max(K-S,0),S\in(0,S^*(T))$.

On the best implementation boundary $S^*(t)$, there are $V(S^*(t),t)=max(K-S^*(t),0)$ and $\frac{\partial V(S^*(t),t)}{\partial S}=-1,t\in(0,T]$, and the boundary conditions are also satisfied $\lim_{s\to\infty}V(S,t)=0$, and $\lim_{s\to 0^+}V(0,t)=Ke^{-r(T-t)}$.[16]

American option pricing is to find the function $V(t,S(t))$ in the holding region after determining the optimal execution boundary, so that it satisfies the above partial differential equation. About the method of solving partial differential equations, the finite difference method is more commonly used, that means the approximate difference equation water is substituted for the Black-Scholes partial differential equation, and then the difference equation is solved by iterative method, the final number obtained is the option price.

3. Advantages and Disadvantages

This model has several obvious advantages and disadvantage. First, simplicity and speed Black-Scholes model provides a relatively simple and understandable formula for the pricing of European call and put options. The formula is based on variables such as the price of the underlying asset, the exercise price, the expiration time, the risk-free interest rate and the volatility, and can be calculated quickly and efficiently. Traders and investors can quickly assess option prices to make timely decisions in fast-paced financial markets. The Black-Scholes model has become the standard for option pricing, providing a widely accepted and consistent method for determining the value of options. This standardization promotes a common framework for options pricing and trading, improving market efficiency. It provides a level playing field for market participants and fosters a transparent and organized options market. In addition, it can also carry out risk management by estimating option prices, Black-Scholes model helps to evaluate and manage the risks associated with options. Traders and investors can analyze the potential exposure of their options positions to help with risk assessment and proper portfolio management. This understanding of risk helps develop strategies to mitigate potential losses and optimize returns [17]. The most obvious is also in academic education as Black-Scholes models serve as an educational tool to introduce students and practitioners to the fundamentals of option pricing and financial mathematics. Its relatively simple formulas and underlying assumptions make it an easy starting point for studying options and derivatives. It lays the foundation for more advanced models and encourages a deeper understanding of financial markets. After developing many different versions of the BS option pricing model (addressing the different assumptions of the model), the use and testing of artificial neural networks (nn) in option pricing has attracted the attention of financial researchers because it is an alternative pricing model that does not require assumptions about variables and their relationships [18].

While there are advantages, there are some drawbacks, such as: the constant fluctuation hypothesis: one of the key assumptions of the Black-Scholes model.

Volatility remains constant over the life of the option. In fact, volatility is dynamic and can change significantly due to various market factors, news events, or economic conditions. This assumption can lead to inaccurate option valuations, especially during periods of high volatility and market efficiency assumptions: The Black-Scholes model is built on the assumption that financial markets are completely efficient and follow a continuous and random price movement (geometric Brownian motion). However, real markets can exhibit deviations from perfect efficiency due to transaction costs, liquidity constraints, and behavioral biases. Ignoring these factors can lead to mispricing of options and unrealistic expectations [19]. Limited applicability: Black-Scholes models are primarily designed for European-style options, which can only be executed at expiration. It does not accurately price American-style options, which can be exercised before expiration. In addition, the model assumes constant interest rates and does not take into account dividends, limiting its direct applicability to certain real-world scenarios, such as dividend-paying stocks or rate-sensitive options. Complexity of inputs: While the Black-Scholes formula may seem simple, it requires accurate and precise inputs, such as volatility estimates and interest rates. As for the time-fractional B-S model (TFBSM)(6), the research on its numerical simulation is relatively new and limited from the existing literature [20]. Estimating these inputs can be challenging, especially for less liquid or newly launched securities, which leads to uncertainty in option pricing. Then there are the unrealistic assumptions: the Black-Scholes model assumes that returns are normal.

4. Conclusion

Fragmented markets are frictionless [21]. These assumptions, while helpful for simplification and mathematical traceability, do not fully match real-world market dynamics. The abnormal nature of market friction, transaction costs and returns are often important factors affecting option prices.

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