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Is Set theory Justified: the Effectiveness and Sufficiency of Set Theory as a Solution to the Mathematical Crisis

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Abstract:

This article presents an objection to the claim that the iterative conception of the set (ICS) is a potential solution to the justification of the foundation of mathematics. The beginning of this article provides an overview of ICS, which is thought to be the paradigm underlying contemporary set theory, and the stage theory, an axiomatized account of ICS. Then, using Boolos' research as support, an argument is made to defend ICS's failure to uphold the axioms of choice, extensionality, and the axiom schema of replacement. After that, a response based on Alexander Paseau's work was given to Boolos' objection, claiming that a second conception of set, the Frege-von Neumann conception (FN), justifies the remaining axioms which ICS failed. This article concludes by concluding that the commonly taken-for-granted set theory is unjustified, calling for people's attention to this significant topic.

Keywords: iterative conception of the set; naive set theory; philosophy of mathematics.

1. Introduction

Set theory serves as the foundation for mathematics and played an important role in many contemporary mathematics developments. However, the justification of set theory has long been neglected [1]. By arguing the insufficiency of the iterative conception of the set (ICS) in terms of justifying set theory axioms and responding to a major objection to that argument, this article questions the widely accepted Zermelo–Fraenkel set theory (ZFC) and claiming it to be not fully justified.

This article deals with some of the theoretical difficulties of the ZFC hypothesis set. Then it turns to possible solutions of FN for unknown axiomatic problems, after which the present study lends Alexander Paseau a further push by arguing that making corrections to the concept of set has already been disproved in the work on FN [2]. Within the limits of this study, these controversial judgments are not simply a repetition of the scholarly position; these arguments attempt to re-suggest to future scholars that the discussion about the foundations of modern mathematics can be answered once more and that the answers that have been given, may be rewritten once more.

2. Exegesis

The iterative conception of the set is an intuitive conception that was popularized by George Boolos in his 1989 paper as a solution to Russell's paradox in the naive set theory [3]. He formalized this conception using an axiomatic theory called stage theory [3, 4]. The rough idea of the ICS is that pure sets are formed in stages. By describing the process of the formation of the set, ICS defines sets as collections formed at any of those stages. The process of formation is as follows:

At stage zero, all possible sets containing any combination of previous elements are formed. Since nothing has yet been formed at stage zero, there will only be one set, namely the null set. Following the 0th stage is stage 1, at which all available collections of sets formed in stage zero, which are the null set and the set containing the null set. Then, at Stage 2, form all available collections of sets formed at Stage 0 and Stage 1. Continuing in this fashion, all available collections of sets formed at earlier stages are formed at every finite stage. Immediately after the finite stages comes stage . At stage , form all available collections of any sets formed at the previous stages. Then comes stage +1, +2, +3 ... Immediately after stage +1, +2, +3... comes stage 1,1+1, 1+2... then comes 2, 3...... A set is a collection formed at some stage of this never-ending process.

It is quite plain that this conception prevents the formation of any self-containing set that would result in Russell's

paradox. Nor are any two sets such that X contains Y and Y contains X allowed to form, or any ‚loops' such that , ... Boolos formalizes the ICS using six axioms, which he calls stage theory. The language used by the stage theory is a first-order logic language with variables standing for the set (x, y, z...) and for stages (r, s, t...) and with three two-place predicate letters, , , F, where is read as ‚is earlier than' and F as ‚is formed at.' The six axioms of the stage theory are:
Tra
(Earlier than is transitive.)
Net
(There is a later stage for all given stages.)
Inf
(There is an infinite stage.)
All
(Every set is formed at some stage.)
When
(A set is formed at a certain stage if all its elements are formed before this stage.)
Spec
(For any specifiable property A, there is a set of all A-sets if all A-sets are formed earlier than some stage.)
Stage theory can be considered an intermediate theory between the informal ICS and the formal ZFC theory, acting as a bridge between them [3, 5].
Argument
The ZFC set theory consists of the axiom of extensionality, union, pairs, powersets, foundation, infinity, and choice, and the axioms of replacement and separation. Even though Boolos managed to derive most of the axioms of ZFC from his stage theory, three axioms are independent of the stage theory, which demonstrates the ICS is insufficient in providing a sound foundation for ZFC.

3. The Axiom of Extensionality

The axiom of extensionality says that their members, i.e. determine the identity of sets, two sets are identical if they have the same members. It might be surprising that the seemingly most non-controversial and fundamental axiom cannot be derived from the ICS. Yet, the theory indeed does not inherently lead to the axiom of extensionality, even while it offers a quite good framework for comprehending sets and their creation through stages.
The iteration conception of sets focuses on building sets through stages. What ICS emphasizes is, as its name suggests, the idea that sets are being formed iteratively and systematically. It does not directly address the concept of set equity based on membership. Ergo, ICS left this axiom unjustified.
Nonetheless, since this axiom of extensionality is so fundamental for set theory and seems so self-evident, one might argue that it is ‚analytic,' that is to say, it is true by the meanings of the words in it. The axiom of extensionality is inevitably true just like how ‚vertebrates have spines' or ‚all mortals will die' are true. Quine and others have made a strong case for the need to hold off on claiming anything analytic until we have a satisfactory explanation for how a statement (or what it says) can be true due to its meanings. Therefore, as long as a persuasive argument regarding the validity of analytic truth has not been made, the axiom of extensionality shall not be considered justified.

3.1 The Axiom of Choice

The second axiom that could not be derived from ICS is the axiom of choice, which states that ‚if there is a set X whose members are disjoint sets, then a set that contains exactly one member of every member of X can be formed'. The axiom of choice is sometimes referred to as the axiom of well-ordering since, in fact, the two are equivalent.
Again, unfortunately, ISC appears to be sitting on the fence concerning the axiom of choice. Both the axiom of choice and its negation can consistently fit under the stage theory; in other words, the axiom of choice is independent of ICS. Being independent means that neither the axiom nor its negation can be proved to be a theorem of ZFC; neither the axiom nor its negation can be derived from ICS.
Consider how one might try to obtain the axiom of choice from ICS: Let X be a nonempty set with disjoint sets as members that formed at some stage S. According to When all members of X are formed before S so that X can be formed at S. Since the members of X are formed, their members must form in stages earlier than S for the members of X to form. Thus, S must form (if not already formed) a set containing exactly one member from every member of X. This derivation might seem valid at first sight. Yet, the problem is that in some cases, it is impossible to pick a member out of each member of X.
Bertrand Russell explained why it might be difficult to choose a member from a set with a vivid analogy: It is feasible to design a choice function directly for any (even infinite) collection of pairs of shoes by selecting the left shoe from each pair to create an appropriate collection (i.e., set) of shoes. Yet, without using the axiom of choice, creating a function that chooses one sock from each pair in an endless collection of socks (assuming they have no distinguishing characteristics) is impossible. Eventually,

we still find ourselves depending on the axiom of choice, and this argument is therefore false, as well as the claim that the axiom of choice is justified.

3.2 The Axiom Schema of Replacement

The axiom schema of replacement in ZFC is a schema of axioms that states that any set‘s image under any definable mapping is also a set. Constructing certain infinite sets in ZFC requires invoking the axiom schema of replacement. This, along with other desirable results due to the axioms of replacement and the absence of undesirable ones, is the main reason the schema is adopted. Yet this does not mean that the axiom schema of replacement can be taken for granted.

The justness of the axioms of replacement relies on specific conditions related to functions and set transformations that are not explicitly addressed in ICS. As pointed out by Boolos, more presumptions or principles would be needed to derive the axioms of replacement, in addition to the iterative conception‘s rough description of sets and stages. For instance, a principle that states, ‚If each set is (in some way) related to at least one stage, then for any set X there is a stage S such that for each member Y of X which is related to stage R, S is a later stage than R‘ would do the job. Even though this further thought describing the correlation between sets and stages seems reasonable and is rather tempting, it is still a further thought and not something that can be said to have been meant in the rough description of ICS. This extension of ICS cannot be dependent on justifying axioms. Therefore, ICS is not a valid reason for us to believe the axiom schema of replacement is justified.

3.3 Objection

Even though Boolos himself demonstrated some of the ZFC axioms being extrinsic or instrumental, he argued for the justness of these axioms in his later work by invoking the second conception in addition to ICS, which he considered to be underpinning contemporary set theory. Blending Frege’s Basic Law V with a size-limiting principle based on Cantor, Russell, von Neumann, and Bernays’ ideas, Boolos named this conception Frege-von Neumann (FN) [5].

While taking ‘there are at least two thoughts “behind” set theory’ as a premise, Boolos argued for the justness of the axioms that ICS failed to by demonstrating how FN does a better job justifying those axioms.

If both VN and ICS are accurate depictions of the set universe, and at least one of the conceptions justifies each of the axioms of ZFC, then there shall be no hesitation and doubt in adopting these axioms. This is how he led himself to think that the theory was justified.

3.4 Responses

Regarding this objection, Paseau made a strong case to answer it. He first identified four criteria Boolos used for an adequate justification: The background conception must be natural and consistent. The background conception must be an actual conception. Choosing whether a principle is part of the background conception must be easier than accepting an axiom. The obviousness of a principle must not be independent from that of the rest of the background conception. Then, he demonstrated that each of Boolos‘s defenses for the axiom of extensionality, choice, and axiom schema of replacement failed to meet at least one of Boolos‘ criteria. Therefore, FN should not be considered a better conception that justifies the three remaining axioms. Since both ICS and FN cannot provide a satisfying justification for the axioms of ZFC, it is better not to consider contemporary set theory as fully justified.

4. Conclusion

In light of the presented argument and thorough examination of the iterative conception of the set (ICS) and Frege-von Neumann (FN) conception within the realm of set theory, it becomes evident that neither of these conceptions fully addresses the justifications for the axioms of Zermelo-Fraenkel set theory (ZFC). While serving as a conventional paradigm, the ICS‘s approach falls short of adequately justifying the foundational axioms of set theory, particularly the axiom of extensionality, the axiom of choice, and the axiom schema of replacement. Similarly, Boolos‘ FN conception, despite its efforts to amalgamate various historical perspectives ranging from Cantor to Bernays, does not meet the criteria Boolos himself laid out for an adequate justification of set theory axioms. They are naturalness and consistency, actual conception, ease of principle inclusion, and interdependence of obviousness with the rest of the background conception. Each of Boolos‘ defenses for the axioms fails to satisfy at least one of these standards, concluding that FN should not be seen as a superior concept for justifying the ZFC axioms.

Therefore, while ICS and FN provide valuable insights into the fabric of set theory‘s justifications, this analysis leads to a sobering reflection that contemporary set theory, according to the criteria and discussions at hand, should not be fully justified. Pursuing justification, intrinsic to the philosophy of mathematics, demands ongoing critical inquiry and thoughtful reflection. Although the insight brought forth by ICS and FN generates progress in understanding, the journey towards a complete philosophical underpinning of set theory remains an open and compelling question. Acknowledging this gap is not an end but rather an invitation for continued exploration and refinement of the principles that seek to explain the foundation upon which mathematics stands. This inquiry draws the mathematics community's attention to the need for rigorous scrutiny when considering the axioms that shape our understanding of set theory and, by extension, the broader mathematical landscape. It is a call to probe deeper into our justifications and remain open to the evolution of thought that might emerge from the complex interplay of logic, philosophy, and mathematics.

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